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ANALYSIS OF THE BOUNDARY CONDITION AT THE INTERFACE BETWEEN A VISCOUS FLUID AND A POROUS MEDIUM AND RELATED MODELLING QUESTIONS

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ABSTRACT. – This paper deals with three fundamental modelling questions for the Darcy and Brinkman equations for flow in a porous medium. It is first shown that in the Dirichlet initial - boundary value problem for the Brinkman equations the solution depends continuously on the viscous coefficient. Then L^2 convergence of the solution of this problem to the solution of an analogous problem for the Darcy equations is established. Finally, it is proved that for flow in a domain occupied by a viscous fluid in contact with a porous solid, the solution depends continuously on a coefficient in the interface boundary condition. The continuous dependence holds for Stokes flow in the fluid, and the analogous Navier-Stokes situation is discussed. © Elsevier, Paris

1. Introduction

There has been much recent interest in obtaining stability estimates for solutions to physical problems in partial differen tial equations where changes in coefficients are allowed, or even the equations themselves change. This type of stability, which is often called structural stability to distinguish it from continuous dependence on the initial data, is studied for example in Ames & Payne (1994a,b; 1997a,b), Bennett (1986, 1991), Franchi & Straughan (1993, 1994a,b, 1996), Lin & Payne (1993, 1994), Morro & Straughan (1992), Payne (1987a,b), Payne & Song (1997) and Payne & Straughan (1989, 1990). Such stability estimates are fundamental in that one wishes to know if a small change in a coefficient in an equation or boundary data, or a small change in the equations themselves, will lead to a drastic change in the solution. To give a concrete example of a structural stability or continuous dependence on modelling question we may refer to the paper of Payne & Straughan (1989) where the difference between a solution to the Stokes equations for slow viscous flow and a solution to the Navier-Stokes equations is investigated. There it is shown how a solution to the Stokes equations approximates a solution to the full Navier-Stokes equations. Thus, questions of continuous dependence on the model itself are fundamental and in many ways are as important as a study of stability itself.

In this paper we investigate three related topics. We investigate how the solution behaves to flow in a domain in which clean fluid borders a porous medium. We establish a continuous dependence result on a crucial parameter which appears in the interfacial

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boundary condition. The equations for flow in the fluid are the well known Darcy (1856) equations; these equations are discussed at length in Nield & Bejan (1992). We also investigate the Brinkman correction to Darcy's equations; this is again explained in depth in Nield & Bejan (1992). We obtain *a priori* results which demonstrate how the solution to the Brinkman system depends continuously on the Brinkman effective viscosity coefficient. We also derive an *a priori* convergence result which compares the solution to the Brinkman system of partial differen tial equations with that of the Darcy equations.

It is worth drawing attention to the fact that the Brinkman equations have been the subject of much recent attention. Nield & Bejan (1992) describe several applications, and work of Guo & Kaloni (1995a,b) and Qin & Kaloni (1993, 1994) concentrates on obtaining quantitative results in physical problems. Payne & Straughan (1996) and Payne & Song (1997) investigate mathematical properties of solutions to both the Darcy and Brinkman equations.

The purpose of this paper is to derive *a priori* continuous dependence estimates. *A priori* estimates in fluid dynamic situations such as those for the Navier-Stokes equations are notoriously difficult to obtain. We are, in a sense, continuing the study of Payne & Straughan (1996) who derived *a priori* estimates for solutions to the initial - time geometry problem for the Brinkman and Darcy equations of flow in porous media.

The equations for flow through porous media under a variety of boundary conditions are reviewed at length in the very clear book of Nield & Bejan (1992). For non-isothermal flow the Darcy equations consist of an equation of momentum for the velocity, v_i,

$$v_i = -\frac{\partial p}{\partial x_i} + g_i T, \tag{1.1}$$

where p, g_i, T are pressure, gravity, and temperature. Standard indicial notation together with the Einstein summation convention is employed throughout. The equation of continuity is

$$v_{i,i} = 0, \tag{1.2}$$

and the balance of energy equation assumes the form

$$\frac{\partial T}{\partial t} + v_i \frac{\partial T}{\partial x_i} = \Delta T. \tag{1.3}$$

Equations (1.1)-(1.3) are Darcy's equations for flow in porous media, accounting for non-constant temperature effects and employing the Boussinesq approximation in the buoyancy term in (1.1). (The thermal diffusivity, viscosity, and permeability have been scaled out since these play no role in the ensuing analysis.) Equations (1.1)-(1.3) are studied on a bounded domain $\Omega \subset \mathbf{R}^3$, with boundary $\partial\Omega$, sufficiently smooth to allow applications of the divergence theorem. The gravity vector is assumed constant and the equations are scaled so that

$$|\mathbf{g}| \leq 1. \tag{1.4}$$

When the porosity of the material (*i.e.* the volume of fluid / total volume) is close to one, or when the flow is near a solid (impermeable) boundary, Nield & Bejan (1992) include physical reasons why equation (1.1) may need to be modified to include an effective viscosity term, viz.

$$v_i - \lambda\Delta v_i = -\frac{\partial p}{\partial x_i} + g_i T. \tag{1.5}$$

In equation (1.5) λ is the (positive) effective viscosity coefficient. Equations (1.2), (1.3) and (1.5) comprise the Brinkman system of partial differen tial equations which are again studied on a bounded domain Ω.

The plan of the paper is as follows. In section 2 we study continuous dependence of a solution to the Brinkman system on the effective viscosity coefficient λ. In section 3 the topic of convergence of a solution to the Brinkman system to a solution to Darcy's equations is investigated. Finally, in section 4 we investigate the continuous dependence of the solution on an interfacial parameter when Stokes flow occurs in a domain which also includes a porous medium. The last section develops an *a priori* result and is particularly interesting in that the analysis involves introduction of a novel functional due to the differences between the Darcy and Stokes flow equations.

2. Continuous dependence on the Brinkman coefficient

In this section we consider the question of continuous dependence on the effective viscosity coefficient λ for a solution to equations (1.2), (1.3) and (1.5). Thus, let (v_i^*, T^*, p^*) satisfy the partial differen tial equations

$$\begin{cases} v_i^* - \lambda^*\Delta v_i^* = -\dfrac{\partial p^*}{\partial x_i} + g_i T^*, \\ v_{i,i}^* = 0, \\ \dfrac{\partial T^*}{\partial t} + v_i^* \dfrac{\partial T^*}{\partial x_i} = \Delta T^*, \end{cases} \tag{2.1}$$

on $\Omega \times (0, \mathcal{T})$, for some positive time $\mathcal{T}$, together with the boundary conditions

$$v_i^* = 0, \qquad T^* = h(x,t), \qquad \text{on} \quad \partial\Omega \times (0, \mathcal{T}), \tag{2.2}$$

and the initial condition

$$T^*(x,0) = T_0(x), \qquad x \in \Omega, \tag{2.3}$$

where h, T_0 are prescribed functions. Further, suppose (v_i, T, p) satisfy the boundary initial value problem

$$\begin{cases} v_i - \lambda\Delta v_i = -\dfrac{\partial p}{\partial x_i} + g_i T, \\ v_{i,i} = 0, \\ \dfrac{\partial T}{\partial t} + v_i \dfrac{\partial T}{\partial x_i} = \Delta T, \end{cases} \tag{2.4}$$

on $\Omega \times (0, \mathcal{T})$,

$$v_i = 0, \qquad T = h(x,t), \qquad \text{on} \quad \partial\Omega \times (0, \mathcal{T}), \tag{2.5}$$

and

$$T(x,0) = T_0(x), \qquad x \in \Omega. \tag{2.6}$$

Define the difference solution (u_i, θ, π) by

$$u_i = v_i^* - v_i, \qquad \theta = T^* - T, \qquad \pi = p^* - p,$$

and define σ by

$$\sigma = \lambda^* - \lambda. \tag{2.7}$$

Then we find the difference solution (u_i, θ, π) satisfies the boundary initial value problem

$$\begin{cases} -\sigma\Delta v_i^* - \lambda\Delta u_i + u_i = -\dfrac{\partial \pi}{\partial x_i} + g_i\theta, \\ u_{i,i} = 0, \\ \dfrac{\partial \theta}{\partial t} + v_i^* \dfrac{\partial \theta}{\partial x_i} + u_i \dfrac{\partial T}{\partial x_i} = \Delta\theta, \end{cases} \tag{2.8}$$

on $\Omega \times (0, \mathcal{T})$,

$$u_i = 0, \qquad \theta = 0, \qquad \text{on} \quad \partial\Omega \times (0, \mathcal{T}), \tag{2.9}$$

$$\theta(x,0) = 0, \qquad x \in \Omega. \tag{2.10}$$

Let $< \cdot >$ denote integration over Ω, and let $\|\cdot\|$ and $(\,,)$ denote the norm and inner product on $L^2(\Omega)$.

The following theorem establishes continuous dependence of the solution to (2.4)-(2.6) on the effective viscosity coefficient λ. Part (a) requires $\partial T/\partial n \in L^1(\partial\Omega)$ whereas this requirement is relaxed in part (b) at the expense of more technicalities.

THEOREM 1. – (a) *If $\partial T/\partial n \in L^1(\partial\Omega)$ or $|\mathbf{v}|$ is bounded in $\Omega \times [0, \mathcal{T}]$, then u_i, θ satisfy the following estimates*

$$\|\theta(t)\|^2 \le F_1(t)\,\frac{\sigma^2}{\lambda\lambda^*},$$

$$\|\mathbf{u}(t)\|^2 \le F_2(t)\,\frac{\sigma^2}{\lambda\lambda^*},$$

$$\|\nabla\mathbf{u}(t)\|^2 \le F_3(t)\,\frac{\sigma^2}{\lambda\lambda^*},$$

where the data functions $F_i(t)$ are given by (2.24) *and* (2.25) *and $t \in (0, \mathcal{T})$.*

(b) *The functions* u_i, θ *satisy the a priori estimates*

$$\|\theta(t)\|^2 \le G_1(t)\,\frac{\sigma^2}{\lambda\lambda^*},$$

$$\|\mathbf{u}(t)\|^2 \le G_2(t)\,\frac{\sigma^2}{\lambda\lambda^*},$$

$$\|\nabla\mathbf{u}(t)\|^2 \le G_3(t)\,\frac{\sigma^2}{\lambda\lambda^*},$$

for $t \in (0, \mathcal{T})$, *and where the functions* $G_i(t)$ *which depend on data are given by* (2.65), (2.66) *and* (2.67).

Proof. – The proofs of parts (a) and (b) have a common beginning.

We commence by multiplying $(2.8)_3$ by θ and integrating over Ω to find with integration by parts

$$\frac{d}{dt}\frac{1}{2}\|\theta\|^2 = < u_i T \theta_{,i} > - \|\nabla\theta\|^2. \tag{2.11}$$

Further, by multiplication of $(2.8)_1$ by u_i and integration we obtain

$$\begin{aligned} \|\mathbf{u}\|^2 + \lambda\|\nabla\mathbf{u}\|^2 &= g_i(\theta, u_i) - \sigma(\nabla v_i^*, \nabla u_i) \\ &\le \|\theta\|\|\mathbf{u}\| + \sigma\|\nabla\mathbf{v}^*\|\|\nabla\mathbf{u}\|, \end{aligned} \tag{2.12}$$

where the Cauchy-Schwarz inequality has also been employed. Upon using the arithmetic-geometric mean inequality in (2.12) we may obtain

$$\|\mathbf{u}\|^2 + \lambda\|\nabla\mathbf{u}\|^2 \le \|\theta\|^2 + \frac{\sigma^2}{\lambda}\|\nabla\mathbf{v}^*\|^2. \tag{2.13}$$

From (2.13) we clearly obtain the bounds

$$\|\nabla\mathbf{u}\|^2 \le \frac{1}{\lambda}\|\theta\|^2 + \frac{\sigma^2}{\lambda^2}\|\nabla\mathbf{v}^*\|^2, \tag{2.14}$$

$$\|\mathbf{u}\|^2 \le \|\theta\|^2 + \frac{\sigma^2}{\lambda}\|\nabla\mathbf{v}^*\|^2. \tag{2.15}$$

(a) We now estimate T in (2.11). Let p be an even integer and then differentiating and using equation $(2.4)_3$ we obtain

$$\begin{aligned} \frac{d}{dt}\int_\Omega T^p dx &= p\int_\Omega T^{p-1}(\Delta T - v_i T_{,i})\,dx \\ &= -p(p-1)\int_\Omega T^{p-2} T_{,i} T_{,i}\,dx + p\oint_{\partial\Omega} T^{p-1}\frac{\partial T}{\partial n}\,dA, \end{aligned} \tag{2.16}$$

after use of $(2.4)_2$ and integration by parts. The negative term is dropped from the right of (2.16) and after integration in t we may obtain:

$$\Big(\int_\Omega T^p dx\Big)^{1/p} \le \Big[\int_\Omega T_0^p dx + p h_m^{p-1}\int_0^t \oint_{\partial\Omega}\Big|\frac{\partial T}{\partial n}\Big|\, dA\, d\eta\Big]^{1/p},$$

where h_m is the maximum value of $|T|$ on $\partial\Omega$. We now use the fact that if $a \ge b \ge 0$,

$$\begin{aligned}(a^p + b^p)^{1/p} &= a\Big[1 + \Big(\frac{b}{a}\Big)^p\Big]^{1/p},\\ &\le 2^{1/p}a,\end{aligned}$$

with

$$a = \Big(\int_\Omega T_0^p dx\Big)^{1/p}, \qquad b = \Big(p h_m^{p-1}\int_0^t \oint_{\partial\Omega}\Big|\frac{\partial T}{\partial n}\Big|\, dA\, d\eta\Big)^{1/p},$$

or

$$(a^p + b^p)^{1/p} \le 2^{1/p} b,$$

if $b \ge a \ge 0$. This leads to

$$\Big(\int_\Omega T^p dx\Big)^{1/p} \le 2^{1/p}\max\Big\{\Big(\int_\Omega T_0^p dx\Big)^{1/p}, \Big(p h_m^{p-1}\int_0^t \oint_{\partial\Omega}\Big|\frac{\partial T}{\partial n}\Big|\, dA\, d\eta\Big)^{1/p}\Big\}. \tag{2.17}$$

Now let $p \to \infty$ in (2.17) to obtain

$$\sup_{\Omega\times[0,\mathcal{T}]} |T| \le R_0, \tag{2.18}$$

where R_0 is the data term

$$R_0 = \max\big\{|T_0|_m,\ \sup_{[0,\mathcal{T}]} h_m\big\} \tag{2.19}$$

and $|T_0|_m$ denotes the maximum of the absolute value of T_0 over Ω.

If v_i is bounded in $\Omega\times[0,\mathcal{T}]$ then (2.18) follows directly from the maximum principle.

Let $|T_m|$ denote the maximum value of $|T|$ over $\Omega\times(0,t)$. From equation (2.11) we obtain

$$\frac{d}{dt}\frac{1}{2}\|\theta\|^2 \le |T_m|\,\|\mathbf{u}\|\|\nabla\theta\| - \|\nabla\theta\|^2.$$

Next employ the arithmetic-geometric mean inequality and estimate (2.15) to arrive at

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|\theta\|^2 &\le \frac{|T_m|^2}{4}\|\mathbf{u}\|^2\\ &\le \frac{|T_m|^2}{4}\|\theta\|^2 + \frac{|T_m|^2}{4\lambda}\sigma^2\|\nabla\mathbf{v}^*\|^2.\end{aligned} \tag{2.20}$$

To estimate the last term in (2.20) we derive from (2.1)

$$\begin{aligned}\lambda^*\|\nabla \mathbf{v}^*\|^2 + \|\mathbf{v}^*\|^2 &= g_i(T^*, v_i^*)\\ &\leq \|T^*\|\|\mathbf{v}^*\|,\end{aligned}$$

from which we see that

$$\lambda^*\|\nabla \mathbf{v}^*\|^2 \leq \frac{1}{4}\|T^*\|^2 \leq \frac{m(\Omega)}{4}|T_m|^2, \tag{2.21}$$

where $m(\Omega)$ denotes the measure of Ω, and $|T_m|$ also denotes the maximum value of $|T^*|$.

Thus, employing (2.21) in (2.20) yields

$$\frac{d}{dt}\frac{1}{2}\|\theta\|^2 \leq \frac{|T_m|^2}{4}\|\theta\|^2 + \frac{|T_m|^4}{16\lambda\lambda^*}m(\Omega)\sigma^2. \tag{2.22}$$

The procedure leading to (2.18) also applies to T^* and so we use estimate (2.18) in (2.22) to deduce

$$\frac{d}{dt}\|\theta\|^2 \leq \frac{1}{2}R_0^2\|\theta\|^2 + \frac{R_0^4 m(\Omega)}{8}\frac{\sigma^2}{\lambda\lambda^*}.$$

Upon integration one finds

$$\|\theta(t)\|^2 \leq F_1(t)\frac{\sigma^2}{\lambda\lambda^*}, \tag{2.23}$$

where

$$F_1(t) = \frac{R_0^2 m(\Omega)}{4}\exp\left[\frac{R_0^2}{2}t\right]. \tag{2.24}$$

Use of (2.21) and (2.18) in (2.14) and (2.15) then leads with the aid of (2.24) to the inequalities

$$\|\nabla \mathbf{u}\|^2 \leq F_3(t)\frac{\sigma^2}{\lambda\lambda^*}$$

and

$$\|\mathbf{u}\|^2 \leq F_2(t)\frac{\sigma^2}{\lambda\lambda^*},$$

where

$$F_2 = F_1(t) + \frac{m(\Omega)R_0^2}{4}, \qquad F_3 = \frac{F_1(t)}{\lambda} + \frac{m(\Omega)R_0^2}{4\lambda}. \tag{2.25}$$

This completes the proof of part (a).

(b) To establish part (b) we commence with (2.11) and use the Cauchy-Schwarz inequality to see that

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|\theta\|^2 &\leq \frac{1}{4}\int_\Omega u_i u_i T^2\,dx\\ &\leq \frac{1}{4}\|\mathbf{u}\|_4^2\,\|T\|_4^2.\end{aligned}$$

We next employ the Sobolev inequality

$$\|\mathbf{u}\|_4^2 \le c_1 \|\mathbf{u}\|^{1/2} \|\nabla \mathbf{u}\|^{3/2} \tag{2.26}$$

to deduce

$$\frac{d}{dt}\|\theta\|^2 \le \frac{1}{2} c_1 \|T\|_4^2 \|\mathbf{u}\|^{1/2} \|\nabla \mathbf{u}\|^{3/2}.$$

The bounds (2.14) and (2.15) are then employed to find

$$\frac{d}{dt}\|\theta\|^2 \le \frac{1}{2\lambda^{3/4}} c_1 \|T\|_4^2 \left(\|\theta\|^2 + \frac{\sigma^2}{\lambda} \|\nabla \mathbf{v}^*\|^2 \right). \tag{2.27}$$

We now proceed to bound $\|T\|_4$.

Introduce the functions $H(x,t)$ and $G(x,t)$ which satisfy

$$\begin{cases} \Delta H = 0, & \text{in } \Omega, \\ H = h^3, & \text{on } \partial\Omega, \end{cases} \tag{2.28}$$

and

$$\begin{cases} \Delta G = 0, & \text{in } \Omega, \\ G = h, & \text{on } \partial\Omega. \end{cases} \tag{2.29}$$

Form the identity

$$\int_0^t \int_\Omega (T^3 - H)(T_{,t} + v_i T_{,i} - \Delta T) dx\, d\eta = 0,$$

by integrations by parts one then finds

$$\begin{aligned} &\frac{1}{4}\|T(t)\|_4^4 + \frac{3}{4}\int_0^t \|\nabla T^2\|^2 d\eta \\ &= \frac{1}{4}\|T_0\|_4^4 + (H,T) - (H_0,T_0) - \int_0^t \int_\Omega H_{,\eta} T\, dx\, d\eta \\ &\quad + \int_0^t \int_\Omega H v_i T_{,i}\, dx\, d\eta + \int_0^t \int_\Omega H_{,i} T_{,i}\, dx\, d\eta. \end{aligned} \tag{2.30}$$

To bound the cubic term on the right of (2.30) we integrate by parts and then use the Cauchy-Schwarz inequality twice, as follows:

$$\begin{aligned} \int_\Omega H v_i T_{,i}\, dx &= -\int_\Omega H_{,i} v_i T\, dx \\ &\le \|\nabla H\| \sqrt{\int_\Omega v_i v_i T^2 dx} \\ &\le \|\nabla H\| \|\mathbf{v}\|_4 \|T\|_4. \end{aligned} \tag{2.31}$$

Let us now observe that $\mathbf{v}$ also satisfies the Sobolev inequality (2.26) so

$$\|\mathbf{v}\|_4 \le c_1^{1/2}\|\mathbf{v}\|^{1/4}\|\nabla\mathbf{v}\|^{3/4}, \tag{2.32}$$

as well as Poincaré's inequality

$$\lambda_1\|\mathbf{v}\|^2 \le \|\nabla\mathbf{v}\|^2, \tag{2.33}$$

for $\lambda_1(\Omega) > 0$. From (2.4) one easily derives

$$\|\mathbf{v}\|^2 + \lambda\|\nabla\mathbf{v}\|^2 \le \|T\|\|\mathbf{v}\|, \tag{2.34}$$

and so we deduce from (2.34) and (2.33),

$$\|\mathbf{v}\| \le \|T\|, \tag{2.35}$$

and

$$\|\nabla\mathbf{v}\| \le \frac{1}{\lambda\sqrt{\lambda_1}}\,\|T\|. \tag{2.36}$$

Upon employing (2.35) and (2.36) in (2.32) we find

$$\|\mathbf{v}\|_4 \le \frac{c_1^{1/2}}{\lambda^{3/4}\lambda_1^{3/8}}\,\|T\|\,. \tag{2.37}$$

Denote by k the constant

$$k = \frac{c_1^{1/2}}{\lambda^{3/4}\lambda_1^{3/8}}$$

and then using (2.37) in (2.31) we deduce

$$\int_\Omega Hv_iT_{,i}\,dx \le k\|\nabla H\|\,\|T\|\,\|T\|_4\,. \tag{2.38}$$

We now use the arithmetic-geometric mean inequality twice to see that

$$\int_\Omega Hv_iT_{,i}\,dx \le \frac{k^4}{4\xi\mu}\,\|\nabla H\|^4 + \frac{\mu}{4\xi}\|T\|_4^4 + \frac{1}{2}\xi\|T\|^2\,, \tag{2.39}$$

for $\xi, \mu > 0$ at our disposal. By now using a Rellich identity, *cf.* Payne & Weinberger (1958), one may show (explicit details are given in Franchi & Straughan (1994), inequality (4.20)):

$$\|\nabla H\|^2 \le J_1(t), \tag{2.40}$$

where

$$J_1(t) = c_2\oint_{\partial\Omega}|\nabla_s h^3|^2 dA, \tag{2.41}$$

where Ω is star-shaped with respect to the origin and c_2 is a constant dependent on the geometry of Ω. Thus, upon integration of (2.39) and use of (2.41) we may see that

$$\begin{aligned}\int_0^t \int_\Omega Hv_i T_{,i}\,dx\,d\eta \leq & \frac{c_2^2 k^4}{4\xi\mu} \int_0^t J_1^2(\eta)d\eta \\ & + \frac{\mu}{4\xi} \int_0^t \|T\|_4^4 d\eta + \frac{1}{2}\xi \int_0^t \|T\|^2 d\eta .\end{aligned} \tag{2.42}$$

We next integrate by parts in the last term in (2.30) using (2.28) to find that

$$\begin{aligned}\int_0^t \int_\Omega H_{,i} T_{,i}\,dx\,d\eta = & \int_0^t \oint_{\partial\Omega} h \frac{\partial H}{\partial n}\,dA\,d\eta \\ \leq & \sqrt{\int_0^t \oint_{\partial\Omega} h^2 dA\,d\eta \int_0^t \oint_{\partial\Omega} \left(\frac{\partial H}{\partial n}\right)^2 dA\,d\eta},\end{aligned} \tag{2.43}$$

where in the last line the Cauchy-Schwarz inequality has been employed. The $\partial H/\partial n$ term is now estimated in terms of boundary data using a Payne-Weinberger (1958) Rellich identity argument, *cf.* Franchi & Straughan (1994) inequality (4.20), to find

$$\int_0^t \int_\Omega H_{,i} T_{,i}\,dx\,d\eta \leq J_2(t), \tag{2.44}$$

where we have defined the data function $J_2(t)$ by

$$J_2(t) = \sqrt{c_2} \sqrt{\int_0^t \oint_{\partial\Omega} h^2 dA\,d\eta \int_0^t \oint_{\partial\Omega} |\nabla_s h^3|^2\,dA\,d\eta}. \tag{2.45}$$

Since T is not identically zero on $\partial\Omega$ the Poincaré inequality for T^2 assumes the form

$$\int_0^t \|\nabla T^2\|^2 d\eta + \alpha_1 \int_0^t \oint_{\partial\Omega} T^4 dA\,d\eta \geq \hat{\lambda}_1 \int_0^t \|T\|_4^4 d\eta, \tag{2.46}$$

for positive constants $\hat{\lambda}_1$ and α_1 dependent on Ω. We now bound the second term on the left of (2.30) with the aid of inequality (2.46), and we employ (2.42) and (2.44) in (2.30). The choice $\xi = 1$ and $\mu = 3\hat{\lambda}_1$ is made and after further use of the arithmetic-geometric mean inequality we may arrive at:

$$\begin{aligned}\frac{1}{4}\|T(t)\|_4^4 \leq & \frac{1}{4}\|T_0\|_4^4 + \frac{3\alpha_1}{4} \int_0^t \oint_{\partial\Omega} h^4 dA\,d\eta + \frac{1}{2}(\|H\|^2 + \|H_0\|^2) \\ & + \frac{1}{2}\|T_0\|^2 + \frac{1}{2} \int_0^t \|H_{,\eta}\|^2 d\eta + J_2(t) + \frac{c_2^2 k^4}{12\hat{\lambda}_1} \int_0^t J_1^2(\eta)d\eta \\ & + \frac{1}{2}\|T(t)\|^2 + \int_0^t \|T\|^2 d\eta.\end{aligned} \tag{2.47}$$

The H terms in (2.47) are bounded using a Rellich identity, *cf.* Payne & Weinberger (1958). In fact, the precise details of the calculation needed here may be found in Franchi & Straughan (1994b), pp. 448, 449. Using this method, constants $k_1, \ldots, k_4$ may be computed in terms of the geometry of Ω, for Ω star shaped, such that

$$
\begin{aligned}
(2.48)\quad \frac{1}{2}(\|H\|^2 + \|H_0\|^2) + \frac{1}{2}\int_0^t \|H_{,\eta}\|^2 d\eta \\
\leq \frac{1}{4}k_1 \oint_{\partial\Omega} h^6 dA + \frac{1}{4}k_2 \oint_{\partial\Omega} |\nabla_s h^3|^2 dA \\
+ \frac{1}{4}k_3 \int_0^t \oint_{\partial\Omega} h^4 h_{,\eta}^2 \, dA\, d\eta + \frac{1}{4}k_4 \int_0^t \oint_{\partial\Omega} |\nabla_s h_{,\eta}^3|^2 \, dA\, d\eta.
\end{aligned}
$$

Now define $D_1 = D_1(\mathcal{T})$ as:

$$
\begin{aligned}
(2.49)\quad D_1(\mathcal{T}) = \max_{[0,\mathcal{T}]} \left\{ k_1 \oint_{\partial\Omega} h^6 dA + k_2 \oint_{\partial\Omega} |\nabla_s h^3|^2 dA \right\} \\
+ k_3 \int_0^{\mathcal{T}} \oint_{\partial\Omega} h^4 h_{,\eta}^2 \, dA\, d\eta + k_4 \int_0^{\mathcal{T}} \oint_{\partial\Omega} |\nabla_s h_{,\eta}^3|^2 \, dA\, d\eta \\
+ 3\alpha_1 \int_0^{\mathcal{T}} \oint_{\partial\Omega} h^4 \, dA\, d\eta + 4J_2(\mathcal{T}) \\
+ \frac{c_2^2 k^4}{3\hat{\lambda}_1} \int_0^{\mathcal{T}} J_1^2(\eta) d\eta.
\end{aligned}
$$

Thus from (2.47)-(2.49) we conclude

$$
(2.50)\qquad \|T(t)\|_4^4 \leq \|T_0\|_4^4 + 2\|T_0\|^2 + D_1 + 2\|T(t)\|^2 + 4\int_0^t \|T\|^2 d\eta.
$$

The next step begins with the identity

$$
(2.51)\qquad \int_0^t \int_\Omega (T - G)(T_{,t} + v_i T_{,i} - \Delta T) dx\, d\eta = 0.
$$

After some manipulation this may be rearranged as:

$$
\begin{aligned}
(2.52)\quad \frac{1}{2}\|T(t)\|^2 + \int_0^t \|\nabla T\|^2 d\eta = \frac{1}{2}\|T_0\|^2 - \int_0^t (G_{,\eta}, T) d\eta + (G, T) - (G_0, T_0) \\
+ \int_0^t < G v_i T_{,i} > d\eta + \int_0^t (\nabla T, \nabla G) d\eta.
\end{aligned}
$$

With the aid of the maximum principle, and by using the arithmetic-geometric mean inequality we may show

$$
\begin{aligned}
(2.53)\qquad \int_0^t \int_\Omega G v_i T_{,i}\, dx\, d\eta &\le h_m \sqrt{\int_0^t \|\mathbf{v}\|^2 d\eta \int_0^t \|\nabla T\|^2 d\eta}\,,\\
&\le h_m \sqrt{\int_0^t \|T\|^2 d\eta \int_0^t \|\nabla T\|^2 d\eta}\,,\\
&\le \frac{h_m}{2\omega}\int_0^t \|T\|^2 d\eta + \frac{h_m \omega}{2}\int_0^t \|\nabla T\|^2 d\eta\,,
\end{aligned}
$$

where (2.35) has also been employed and $\omega > 0$ is a constant to be selected. Thus, employing (2.53) and making further use of the arithmetic-geometric mean inequality in (2.52) we may derive

$$
\begin{aligned}
(2.54)\qquad \frac{1}{2}\|T(t)\|^2 + \int_0^t \|\nabla T\|^2 d\eta \le & \|T_0\|^2 + \|G\|^2 + \frac{1}{2}\|G_0\|^2 + \frac{1}{2}\int_0^t \|G_{,\eta}\|^2 d\eta\\
&+ \frac{1}{4}\|T(t)\|^2 + \frac{h_m\omega}{2}\int_0^t \|\nabla T\|^2 d\eta\\
&+ \frac{1}{2}\Big(1 + \frac{h_m}{\omega}\Big)\int_0^t \|T\|^2 d\eta\\
&+ \int_0^t \oint_{\partial\Omega} h\,\frac{\partial G}{\partial n}\, dA\, d\eta\,,
\end{aligned}
$$

where the last term in (2.52) has been written as a boundary term observing G is harmonic in Ω. Pick now $\omega = 2/h_m$ and employ the Cauchy-Schwarz inequality on the last term in (2.54) to derive

$$
\begin{aligned}
(2.55)\qquad \frac{1}{4}\|T(t)\|^2 \le & \Big(\frac{1}{2} + \frac{h_m^2}{4}\Big)\int_0^t \|T\|^2 d\eta + \|T_0\|^2\\
&+ \|G\|^2 + \frac{1}{2}\|G_0\|^2 + \frac{1}{2}\int_0^t \|G_{,\eta}\|^2 d\eta\\
&+ \sqrt{\int_0^t \oint_{\partial\Omega} h^2 dA\, d\eta \int_0^t \oint_{\partial\Omega} \Big(\frac{\partial G}{\partial n}\Big)^2 dA\, d\eta}\,.
\end{aligned}
$$

Further use of a Rellich identity allows us to bound the last four terms on the right of (2.55) in terms of the boundary data h. In fact, we may determine constants $h_1, \ldots, h_5$, dependent on the geometry of Ω, such that

$$(2.56)\quad \frac{1}{4}D_2(t) = h_1 \oint_{\partial\Omega} h^2 dA + h_2 \oint_{\partial\Omega} |\nabla_s h|^2 dA + h_3\sqrt{\int_0^t \oint_{\partial\Omega} h^2 dA\, d\eta \int_0^t \oint_{\partial\Omega} |\nabla_s h|^2 dA\, d\eta} + h_4 \int_0^t \oint_{\partial\Omega} h_{,\eta}^2 dA\, d\eta + h_5 \int_0^t \oint_{\partial\Omega} |\nabla_s h_{,\eta}|^2 dA\, d\eta,$$

is an upper bound for the G terms on the right of inequality (2.55). It is worth observing that the term $D_2(t)$ is an explicit data term. Thus, put $D_3 = 4\|T_0\|^2 + D_2(\mathcal{T})$, with the understanding that the first two terms in D_2 are maximised over $[0, \mathcal{T}]$. We may then obtain from (2.55) the inequality

$$(2.57)\qquad \|T(t)\|^2 \leq D_3 + a \int_0^t \|T\|^2 d\eta,$$

where we have put

$$a = 2 + h_m^2.$$

Clearly (2.57) leads to

$$\frac{d}{dt}\left(e^{-at}\int_0^t \|T\|^2 d\eta\right) \leq e^{-at} D_3.$$

Upon integration we find

$$(2.58)\qquad \int_0^t \|T\|^2 d\eta \leq \left(\frac{e^{at}-1}{a}\right) D_3.$$

Then, from (2.57) it follows that

$$(2.59)\qquad \|T(t)\|^2 \leq e^{at} D_3.$$

Estimates (2.58) and (2.59) are now employed in (2.50) to yield an *a priori* bound for $\|T\|_4$, namely

$$(2.60)\qquad \|T(t)\|_4^4 \leq D_4^2,$$

where

$$(2.61)\qquad D_4^2 = \|T_0\|_4^4 + 2\|T_0\|^2 + D_1 + 2\big[e^{at} + 2a^{-1}(e^{at}-1)\big] D_3.$$

The bound (2.60) is next used in (2.27) to find

$$\frac{d}{dt}\|\theta\|^2 \leq \frac{c_1}{2\lambda^{3/4}}\, D_4\left(\|\theta\|^2 + \frac{\sigma^2}{\lambda}\|\nabla \mathbf{v}^*\|^2\right). \tag{2.62}$$

Combining (2.21) and (2.59) applied to T^*, we have

$$\|\nabla \mathbf{v}^*\|^2 \leq \frac{1}{4\lambda^*}\,\|T^*\|^2 \leq \frac{e^{at}}{4\lambda^*}\, D_3. \tag{2.63}$$

Thus from (2.62) we deduce

$$\frac{d}{dt}\|\theta\|^2 \leq \frac{c_1 D_4}{2\lambda^{3/4}}\,\|\theta\|^2 + \frac{c_1 D_4 e^{at}}{8\lambda^{3/4}}\,\frac{\sigma^2}{\lambda\lambda^*}. \tag{2.64}$$

An integration of this inequality allows us to derive the *a priori* continuous dependence on λ estimate

$$\|\theta(t)\|^2 \leq \frac{k}{4}\left(\frac{e^{at} - e^{kt}}{a-k}\right)\frac{\sigma^2}{\lambda\lambda^*}, \tag{2.65}$$

where

$$k = \frac{c_1 \bar{D}_4}{2\lambda^{3/4}},$$

with $\bar{D}_4$ denoting D_4 evaluated for $t = \mathcal{T}$. Continuous dependence in the measures $\|\mathbf{u}\|^2$ and $\|\nabla \mathbf{u}\|^2$ may also be derived using (2.14), (2.15), (2.63) and (2.65), to find

$$\|\mathbf{u}\|^2 \leq \frac{1}{4}\left[e^{at} D_3 + k\left(\frac{e^{at} - e^{kt}}{a-k}\right)\right]\frac{\sigma^2}{\lambda\lambda^*} \tag{2.66}$$

and

$$\|\nabla \mathbf{u}\|^2 \leq \frac{1}{4\lambda}\left[e^{at} D_3 + k\left(\frac{e^{at} - e^{kt}}{a-k}\right)\right]\frac{\sigma^2}{\lambda\lambda^*}. \tag{2.67}$$

The proof of Theorem 1 is complete.

Remark. – Let us observe that with the choice $\omega = h_m^{-1}$ in (2.54) instead of (2.55) we may produce

$$\begin{aligned}\frac{1}{4}\|T(t)\|^2 + \frac{1}{2}\int_0^t \|\nabla T\|^2 d\eta \leq{}& \frac{1}{2}(1 + h_m^2)\int_0^t \|T\|^2 d\eta + \|T_0\|^2 \\ &+ \|G\|^2 + \frac{1}{2}\|G_0\|^2 + \frac{1}{2}\int_0^t \|G_{,\eta}\|^2 d\eta \\ &+ \sqrt{\int_0^t \oint_{\partial\Omega} h^2 dA\, d\eta \int_0^t \oint_{\partial\Omega}\left(\frac{\partial G}{\partial n}\right)^2 dA\, d\eta}\,.\end{aligned} \tag{2.68}$$

The steps leading to (2.57) may be repeated, *mutatis mutandis,* and instead of (2.57) one deduces

$$\|T(t)\|^2 + 2\int_0^t \|\nabla T\|^2 d\eta \le D_3 + \hat{a}\int_0^t \|T\|^2 d\eta, \tag{2.69}$$

where

$$\hat{a} = 2 + 2h_m^2. \tag{2.70}$$

Thus, employing the equivalent of (2.58) one finds the bound

$$\int_0^t \|\nabla T\|^2 d\eta \le \frac{1}{2} e^{\hat{a}t} D_3. \tag{2.71}$$

This is useful in the next section.

3. Convergence of a solution to Brinkman's equations to a solution to the Darcy equations

In the present section we derive conditions which ensure a solution to the Brinkman equations will converge in an L^2 sense to a solution to the equations for Darcy flow.

Let (v_i, T, p) be a solution to the Brinkman equations and let (u_i, S, q) be a solution to Darcy's equations, so that

$$\begin{cases} v_i - \lambda\Delta v_i = -\dfrac{\partial p}{\partial x_i} + g_i T, \\ v_{i,i} = 0, \\ \dfrac{\partial T}{\partial t} + v_i \dfrac{\partial T}{\partial x_i} = \Delta T, \end{cases} \tag{3.1}$$

in $\Omega \times (0, \mathcal{T})$ and

$$\begin{cases} u_i = -\dfrac{\partial q}{\partial x_i} + g_i S, \\ u_{i,i} = 0, \\ \dfrac{\partial S}{\partial t} + u_i \dfrac{\partial S}{\partial x_i} = \Delta S, \end{cases} \tag{3.2}$$

in $\Omega \times (0, \mathcal{T})$. On the boundary $\partial\Omega$ we assume

$$v_i = 0, \quad u_i n_i = 0, \quad T = h, \quad S = h, \quad \text{on } \partial\Omega \times (0, \mathcal{T}), \tag{3.3}$$

and at $t = 0$ we require

$$T(x, 0) = T_0(x), \qquad S(x, 0) = T_0(x). \tag{3.4}$$

Define the difference solution (w_i, θ, π) by

$$w_i = u_i - v_i, \quad \theta = S - T, \quad \pi = q - p. \tag{3.5}$$

Then this solution satisfies the boundary initial value problem

$$\begin{cases} w_i + \lambda(v_{i,j} - v_{j,i})_{,j} + \pi_{,i} = g_i\theta, \\ w_{i,i} = 0, \\ \dfrac{\partial\theta}{\partial t} + u_i\theta_{,i} + w_i T_{,i} = \Delta\theta, \end{cases} \tag{3.6}$$

on $\Omega \times (0, \mathcal{T})$,

$$w_i n_i = 0, \quad \theta = 0, \qquad \text{on } \partial\Omega \times (0, \mathcal{T}), \tag{3.7}$$

$$\theta(x, 0) = 0, \qquad x \in \Omega. \tag{3.8}$$

The following theorem establishes convergence of a solution.

THEOREM 2. – *Let* $\partial T/\partial n \in L^1(\partial\Omega \times (0, \mathcal{T}))$. *Then*

$$\int_0^t \|\mathbf{w}\|^2 d\eta \leq J_1(t)\sqrt{\lambda}, \tag{3.9}$$

and

$$\|\theta(t)\|^2 \leq J_2(t)\sqrt{\lambda}, \tag{3.10}$$

where J_1, J_2 *are data functions given by* (3.32) *and* (3.33).

Proof. – Multiply $(3.6)_1$ by w_i and integrate over $\Omega \times (0, t)$ to find with the aid of the divergence theorem, some rearrangement, the solenoidal character of u_i, v_i, and the boundary conditions on u_i, v_i,

$$\int_0^t \int_\Omega w_i \big[w_i + \lambda(v_{i,j} - v_{j,i})_{,j} + \pi_{,i} - g_i\theta\big] \, dx \, d\eta = 0,$$

then

$$\begin{aligned} \int_0^t \|\mathbf{w}\|^2 d\eta + \lambda \int_0^t \oint_{\partial\Omega} u_i n_j (v_{i,j} - v_{j,i}) dA \, d\eta \\ - \lambda \int_0^t \int_\Omega (v_{i,j} - v_{j,i}) w_{i,j} dx \, d\eta - \int_0^t \int_\Omega g_i w_i \theta \, dx \, d\eta = 0, \end{aligned} \tag{3.11}$$

or since $w_i = u_i - v_i$,

$$(3.12)\qquad \int_0^t \|\mathbf{w}\|^2 d\eta + \lambda \int_0^t \int_\Omega (w_{i,j} - w_{j,i}) w_{i,j} dx\, d\eta$$
$$= \lambda \int_0^t \int_\Omega (u_{i,j} - u_{j,i}) w_{i,j} dx\, d\eta - \lambda \int_0^t \oint_{\partial\Omega} u_i \frac{\partial v_i}{\partial n}\, dA\, d\eta$$
$$+ g_i \int_0^t (w_i, \theta)\, d\eta.$$

In deriving (3.12) we have used the fact that on $\partial\Omega$

$$v_{j,i} = n_i \frac{\partial v_j}{\partial n} + a^{\alpha\beta} x^m_{;\alpha} v_{j;\beta} \delta_{mi},$$

where $a^{\alpha\beta}$ is determined from the surface metric tensor. Thus

$$\oint_{\partial\Omega} u_i n_j v_{j,i} dA = \oint_{\partial\Omega} u_i n_i n_j \frac{\partial v_j}{\partial n}\, dA + \oint_{\partial\Omega} a^{\alpha\beta} x^m_{;\alpha} u_m n^j v_{j;\beta}\, dA.$$

The first term on the right is zero since $u_i n_i = 0$ on $\partial\Omega$. Furthermore, since $v_j = 0$ on $\partial\Omega$, $v_{j;\beta} = 0$ on $\partial\Omega$ and so it follows that

$$\oint_{\partial\Omega} u^i n^j v_{j,i} dA = 0$$

and this allows us to drop a term from (3.11).

To bound the boundary term in (3.12) we note that by the Cauchy-Schwarz inequality

$$(3.13)\qquad \int_0^t \oint_{\partial\Omega} u_i \frac{\partial v_i}{\partial n}\, dA\, d\eta \leq \sqrt{\int_0^t \oint_{\partial\Omega} u_i u_i dA\, d\eta \int_0^t \oint_{\partial\Omega} \frac{\partial v_i}{\partial n} \frac{\partial v_i}{\partial n}\, dA\, d\eta}\,.$$

To develop a bound for the right hand side of (3.13) we next introduce a Rellich identity for v_i, viz.

$$(3.14)\qquad \int_0^t \int_\Omega x_j v_{i,j} (v_i - \lambda \Delta v_i + p_{,i} - g_i T) dx\, d\eta = 0.$$

The pressure term in (3.14) integrates to zero. To see this, integrate by parts

$$\int_\Omega x_j v_{i,j} p_{,i} dx = -\int_\Omega v_{i,i} p\, dx + \oint_{\partial\Omega} n_i x^j v^i_{,j} p\, dA$$
$$= \oint_{\partial\Omega} p x^j (n_i v^i_{,j} - n_j v^i_{,i}) dA$$
$$= \oint_{\partial\Omega} p x^j n_i \Big(n_j \frac{\partial v^i}{\partial n} + a^{\alpha\beta} x_{j;\alpha} v^i_{;\beta} \Big) dA$$
$$- \oint_{\partial\Omega} p x^j n_j \Big(n_i \frac{\partial v^i}{\partial n} + a^{\alpha\beta} x_{i;\alpha} v^i_{;\beta} \Big) dA$$
$$= \oint_{\partial\Omega} p a^{\alpha\beta} x^j \big(n_i x_{j;\alpha} v^i_{;\beta} - n_j x_{i;\alpha} v^i_{;\beta} \big) dA$$
$$= 0,$$

the last integrals being comprised of tangential derivatives of v_i. With further integrations by parts (3.14) yields

$$\frac{3}{2}\int_0^t \|\mathbf{v}\|^2 d\eta + \frac{1}{2}\lambda \int_0^t \oint_{\partial\Omega} x^i n_i \frac{\partial v_j}{\partial n}\frac{\partial v_j}{\partial n}\, dA\, d\eta + \frac{1}{2}\lambda \int_0^t \|\nabla \mathbf{v}\|^2 d\eta = g_i \int_0^t \int_\Omega (x_j T_{,j} + 3T) v_i dx\, d\eta.$$

Use of the arithmetic-geometric mean inequality on the right of this then leads to

$$\frac{1}{2}\lambda \int_0^t \|\nabla \mathbf{v}\|^2 d\eta + \frac{1}{2}\lambda \int_0^t \oint_{\partial\Omega} x^i n_i \frac{\partial v_j}{\partial n}\frac{\partial v_j}{\partial n}\, dA\, d\eta \leq \frac{x_m^2}{3}\int_0^t \|\nabla T\|^2 d\eta + 3\int_0^t \|T\|^2 d\eta, \tag{3.15}$$

where x_m is the maximum distance from the origin O to the boundary $\partial\Omega$. Thus for Ω star shaped, say

$$x_i n_i \geq \lambda_0 > 0 \qquad \text{on } \partial\Omega,$$

we have

$$\frac{\lambda\lambda_0}{2}\int_0^t \oint_{\partial\Omega} \frac{\partial v_j}{\partial n}\frac{\partial v_j}{\partial n}\, dA\, d\eta \leq \frac{x_m^2}{3}\int_0^t \|\nabla T\|^2 d\eta + 3\int_0^t \|T\|^2 d\eta. \tag{3.16}$$

Moreover, since Ω is star shaped

$$\begin{aligned}\lambda_0 \oint_{\partial\Omega} u_i u_i dA &\leq \oint_{\partial\Omega} x_j n_j u_i u_i dA \\ &= \int_\Omega (u_i u_i x_j)_{,j} dx \\ &= 3\|\mathbf{u}\|^2 + 2\int_\Omega x_j u_i u_{i,j} dx + 2\int_\Omega x_j u_j u_{i,i} dx,\end{aligned}$$

where we have added a zero term. We next integrate by parts in the last term to obtain

$$\begin{aligned}\lambda_0 \oint_{\partial\Omega} u_i u_i dA &\leq \|\mathbf{u}\|^2 + 2\int_\Omega x_j u_i (u_{i,j} - u_{j,i}) dx \\ &\leq (1 + x_m^2)\|\mathbf{u}\|^2 + \int_\Omega (u_{i,j} - u_{j,i})(u_{i,j} - u_{j,i}) dx.\end{aligned} \tag{3.17}$$

From equation $(3.2)_1$, using the boundary conditions on u_i we find

$$\|\mathbf{u}\|^2 = g_i(S, u_i)$$

from whence it follows that

$$\|\mathbf{u}\| \leq \|S\|. \tag{3.18}$$

Also from $(3.2)_1$,

$$u_{i,j} - u_{j,i} = g_i S_{,j} - g_j S_{,i}$$

and so

$$\int_\Omega (u_{i,j} - u_{j,i})(u_{i,j} - u_{j,i}) dx \leq 4\|\nabla S\|^2. \tag{3.19}$$

Thus (3.18) and (3.19) may be utilized in (3.17) to arrive at

$$\oint_{\partial\Omega} u_i u_i dA \leq \left(\frac{1 + x_m^2}{\lambda_0}\right)\|S\|^2 + \frac{4}{\lambda_0}\|\nabla S\|^2. \tag{3.20}$$

Therefore, we may use (3.16) and (3.20) in (3.13) to deduce

$$\begin{aligned} \int_0^t \oint_{\partial\Omega} u_i \frac{\partial v_i}{\partial n} dA\, d\eta \leq \frac{1}{\sqrt{\lambda}} & \left(\frac{(1 + x_m^2)}{\lambda_0} \int_0^t \|S\|^2 d\eta + \frac{4}{\lambda_0}\int_0^t \|\nabla S\|^2 d\eta\right)^{1/2} \\ & \times \left(\frac{6}{\lambda_0}\int_0^t \|T\|^2 d\eta + \frac{2x_m^2}{3\lambda_0}\int_0^t \|\nabla T\|^2 d\eta\right)^{1/2}. \end{aligned} \tag{3.21}$$

We next show how to bound the T and S terms in (3.21) in terms of data. In fact, the proof leading to (2.58) and (2.71) still holds in this section, hence we may assert that

$$\int_0^t \|T\|^2 d\eta \leq C_1, \qquad \int_0^t \|\nabla T\|^2 d\eta \leq C_2, \tag{3.22}$$

where C_1, C_2 are defined by

$$C_1 = \left(\frac{e^{at} - 1}{a}\right) D_3, \qquad C_2 = \frac{1}{2} e^{\hat{a}t} D_3,$$

with $a, \hat{a}, D_3$ being as defined in section 2. Also, the proof in section 2 leading to (2.58) and (2.71) carries over to $\int_0^t \|S\|^2 d\eta$ and $\int_0^t \|\nabla S\|^2 d\eta$, on utilizing inequality (3.18).

Thus, estimates (3.22) also hold with T replaced by S. Upon using (3.22) and the equivalent estimates for S in (3.21) we may derive

$$\left|\int_0^t \oint_{\partial\Omega} u_i \frac{\partial v_i}{\partial n} dA\, d\eta\right| \leq \frac{D_4}{\sqrt{\lambda}}, \tag{3.23}$$

where the data term D_4 is given by

$$D_4 = \frac{1}{\lambda_0}\left[6(1 + x_m^2)C_1^2 + \left\{24 + \frac{2}{3}x_m^2(1 + x_m^2)\right\}C_1 C_2 + \frac{8x_m^2}{3}C_2^2\right]^{1/2}.$$

Hence, upon returning to (3.12) we find with the aid of (3.23) the inequality

$$\begin{aligned}(3.24)\qquad \int_0^t \|\mathbf{w}\|^2 d\eta + \lambda \int_0^t \int_\Omega (w_{i,j} - w_{j,i}) w_{i,j} dx\, d\eta \\ \leq \lambda \int_0^t \int_\Omega (u_{i,j} - u_{j,i}) w_{i,j} dx\, d\eta + D_4\sqrt{\lambda} \\ + g_i \int_0^t (w_i, \theta)\, d\eta.\end{aligned}$$

To estimate the first term on the right of (3.24) we write $w_{i,j}$ as the sum of its symmetric and skew-symmetric parts so that

$$\begin{aligned}(3.25)\qquad 2\lambda \int_0^t \int_\Omega (u_{i,j} - u_{j,i}) w_{i,j}\, dx\, d\eta = \lambda \int_0^t \int_\Omega (u_{i,j} - u_{j,i})(w_{i,j} - w_{j,i}) dx\, d\eta \\ \leq \frac{1}{2}\lambda \int_0^t \int_\Omega (u_{i,j} - u_{j,i})(u_{i,j} - u_{j,i}) dx\, d\eta \\ + \frac{1}{2}\lambda \int_0^t \int_\Omega (w_{i,j} - w_{j,i})(w_{i,j} - w_{j,i}) dx\, d\eta\,.\end{aligned}$$

Using inequality (3.19) we next find

$$\begin{aligned}(3.26)\qquad & 2\lambda \int_0^t \int_\Omega (u_{i,j} - u_{j,i}) w_{i,j}\, dx\, d\eta \\ & \leq \frac{1}{2}\lambda \int_0^t \int_\Omega (w_{i,j} - w_{j,i})(w_{i,j} - w_{j,i}) dx\, d\eta + 2\lambda \int_0^t \|\nabla S\|^2 d\eta, \\ & \leq \frac{1}{2}\lambda \int_0^t \int_\Omega (w_{i,j} - w_{j,i})(w_{i,j} - w_{j,i}) dx\, d\eta + 2\lambda C_2,\end{aligned}$$

where (3.22) has been used for S. Hence, with (3.26) used in (3.24) we see that

$$(3.27)\quad \int_0^t \|\mathbf{w}\|^2 d\eta + \frac{1}{2}\lambda \int_0^t \int_\Omega (w_{i,j} - w_{j,i}) w_{i,j} dx\, d\eta \leq D_4\sqrt{\lambda} + 2\lambda C_2 + g_i \int_0^t (w_i, \theta)\, d\eta.$$

Next, the arithmetic-geometric mean inequality is employed on the last term in (3.27) as

$$g_i \int_0^t (w_i, \theta) d\eta \leq \frac{1}{2} \int_0^t \|\mathbf{w}\|^2 d\eta + \frac{1}{2} \int_0^t \|\theta\|^2 d\eta.$$

Hence, (3.27) leads to

$$(3.28)\quad \int_0^t \|\mathbf{w}\|^2 d\eta + \lambda \int_0^t \int_\Omega (w_{i,j} - w_{j,i}) w_{i,j} dx\, d\eta \leq 2D_4\sqrt{\lambda} + 4\lambda C_2 + \int_0^t \|\theta\|^2 d\eta.$$

It remains to bound the last term in (3.28) and to this end we multiply $(3.6)_3$ by θ to derive

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 = -\|\nabla\theta\|^2 + \int_\Omega T w_i \theta_{,i} dx \le \frac{R_0^2}{4}\|\mathbf{w}\|^2, \tag{3.29}$$

where R_0 is the data term defined in (2.19) and the proof there is employed to bound T pointwise in the term $(Tw_i, \theta_{,i})$.

Thus, after use of (3.29) in (3.28) we derive

$$\int_0^t \|\mathbf{w}\|^2 d\eta + \lambda \int_0^t \int_\Omega (w_{i,j} - w_{j,i}) w_{i,j} dx\, d\eta \le 2D_4\sqrt{\lambda} + 4\lambda C_2 + \frac{R_0^2}{2}\int_0^t (t-\eta)\|\mathbf{w}\|^2 d\eta. \tag{3.30}$$

Define now $Q = 2D_4 + 4C_2\sqrt{\lambda}$ and set $b = R_0^2/2$. Then we derive from (3.30)

$$\frac{d}{dt}\left(e^{-bt}\int_0^t (t-\eta)\|\mathbf{w}\|^2 d\eta\right) \le e^{-bt} Q\sqrt{\lambda};$$

then by integration,

$$\int_0^t (t-\eta)\|\mathbf{w}\|^2 d\eta \le \frac{Q}{b}\sqrt{\lambda}\left[e^{bt} - 1\right]. \tag{3.31}$$

Also, from (3.30) one may then deduce

$$\int_0^t \|\mathbf{w}\|^2 d\eta \le Q e^{bt}\sqrt{\lambda}, \tag{3.32}$$

and then from (3.29) we have

$$\|\theta(t)\|^2 \le Q b e^{bt}\sqrt{\lambda}. \tag{3.33}$$

Estimates (3.31)-(3.33) establish theorem 2 on convergence of a solution of the Brinkman equations to a solution of Darcy's equations.

Remarks. – 1. We have required the condition $\partial T/\partial n \in L^1(\partial\Omega \times (0, \mathcal{T}))$ in Theorem 2 only in deriving the estimate (3.29). One can avoid imposing this requirement although the proof becomes longer. Details of this calculation are given in the appendix.

2. While Theorem 2 establishes convergence in L^2, pointwise convergence cannot be expected since in general the tangential components of the velocity in the Darcy equations will not vanish on the boundary.

4. Continuous dependence on the interface coefficient

The purpose of this section is to study the manner in which a solution to flow in a fluid which borders a porous medium depends on a coefficient in the interface boundary conditions. Thus, let an appropriate part of the plane $z = x_3 = 0$ denote the boundary between a porous medium occupying a bounded region D_2 in $\mathbf{R}^3$, and a linear viscous fluid occupying a bounded region D_1 in $\mathbf{R}^3$, as shown in figure 1.

The interface is denoted by L while the remaining parts of the boundaries of D_1 and D_2 are denoted, respectively, by Γ_1 and Γ_2. In D_1 we suppose the fluid velocity is slow so the governing equations may be taken to be those of Stokes flow, although the question of Navier-Stokes flow is addressed in a remark at the end of this section. In the porous region D_2 the flow is assumed to satisfy the Darcy (1856) equations.

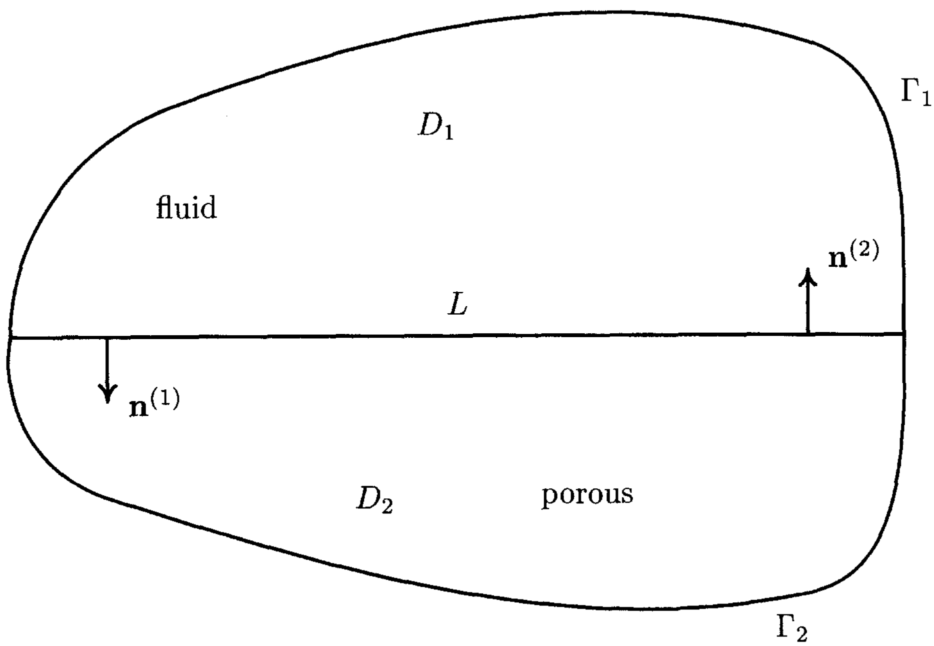

Fig. 1. – Configuration of the fluid overlying a porous layer.

Let (u_i, T, p) and (u_i^m, T^m, p^m) denote the velocity, temperature and pressure in D_1 and D_2, respectively. Then the appropriate Stokes flow equations are

$$(4.1)\qquad \begin{cases} \dfrac{\partial u_i}{\partial t} = -\dfrac{\partial p}{\partial x_i} + \mu \Delta u_i + g_i T, \\ \dfrac{\partial u_i}{\partial x_i} = 0\,, \\ \dfrac{\partial T}{\partial t} + u_i \dfrac{\partial T}{\partial x_i} = \kappa \Delta T, \end{cases}$$

in $D_1 \times (0, \mathcal{T})$, where μ is the dynamic viscosity, κ is the thermal diffusivity and g_i is the gravity vector. We assume (4.1) is scaled so that

$$|\mathbf{g}| \leq 1,$$

and without loss of generality the constant density is taken to have value one.

The relevant Darcy equations are, *cf.* Nield & Bejan (1992):

$$\left\{\begin{aligned} &\frac{\mu}{k} u_i^m = -\frac{\partial p^m}{\partial x_i} + g_i T^m, \\ &\frac{\partial u_i^m}{\partial x_i} = 0, \\ &\frac{\partial T^m}{\partial t} + u_i^m \frac{\partial T^m}{\partial x_i} = \kappa^m \Delta T^m, \end{aligned}\right. \tag{4.2}$$

in $D_2 \times (0, \mathcal{T})$; here k is the permeability and κ^m is the thermal diffusivity of the porous medium.

The functions u_i, T and T^m are required to satisfy the initial data

$$\left\{\begin{aligned} &u_i(x,0) = f_i(x), \qquad T(x,0) = T_0(x), \quad x \in D_1, \\ &T^m(x,0) = T_0^m(x), \quad x \in D_2. \end{aligned}\right. \tag{4.3}$$

On the boundary $\Gamma_1 \cup \Gamma_2$ it is supposed that

$$\left\{\begin{aligned} &u_i = 0, \quad T = T_U(x,t), \qquad \text{on} \quad \Gamma_1 \times (0, \mathcal{T}), \\ &u_i^m n_i = 0, \quad T^m = T_L(x,t), \qquad \text{on} \quad \Gamma_2 \times (0, \mathcal{T}), \end{aligned}\right. \tag{4.4}$$

for prescribed functions T_U and T_L, with n_i being the unit outward normal. The conditions on the interface L are

$$\left\{\begin{aligned} &u_3 = u_3^m, \qquad T = T^m, \qquad \kappa T_{,3} = \kappa^m T_{,3}^m, \\ &p^m = p - 2\mu u_{3,3}, \qquad u_{\beta,3} + u_{3,\beta} = \frac{\alpha_1}{\sqrt{k}} u_\beta, \end{aligned}\right. \tag{4.5}$$

where α_1 is a coefficient determined by experiment for a given fluid and a given porous solid. These boundary conditions are discussed at length in Nield & Bejan (1992). The last condition in (4.5) essentially derives from the work of Jones (1973). Beavers & Joseph (1967) argued on the basis of experimental results that

$$u_{\beta,3} = \frac{\alpha_1}{\sqrt{k}} (u_\beta - u_\beta^m), \tag{4.6}$$

and Jones (1973) generalised this to include the shear stress at the interface, viz.

$$u_{\beta,3} + u_{3,\beta} = \frac{\alpha_1}{\sqrt{k}} (u_\beta - u_\beta^m). \tag{4.7}$$

Nield & Bejan (1992) show that the last term may essentially be dropped and this leads to $(4.5)_5$. It is worth pointing out that condition (4.6) has been employed in a thermal convection study by Nield (1977) and in flow past a sphere by Qin & Kaloni (1993). McKay & Straughan (1993) investigated use of (4.6) and (4.7) in their study of the geophysical problem of formation of stones into regular hexagonal patterns in shallow alpine lakes.

The object of this paper is to derive an *a priori* estimate showing how (u_i, T) and (u_i^m, T^m) depend continuously on the interface coefficient α_1. Thus, let (u_i, p, T) and (u_i^m, p^m, T^m) satisfy (4.1)-(4.5) and let (v_i, q, S) and (v_i^m, q^m, S^m) be a solution to (4.1)-(4.5) with identical data functions f_i, T_0, T_0^m, T_U and T_L, but with α_1 replaced by α_2. Define the difference variables (w_i, π, θ) and σ by

$$w_i = u_i - v_i, \qquad \pi = p - q, \qquad \theta = T - S, \qquad \sigma = \alpha_1 - \alpha_2. \tag{4.8}$$

From (4.1)-(4.5) we find that (w_i, π, θ) satisfy the partial differential equations

$$\begin{cases} \dfrac{\partial w_i}{\partial t} = -\dfrac{\partial \pi}{\partial x_i} + \mu\Delta w_i + g_i\theta, \\ \dfrac{\partial w_i}{\partial x_i} = 0, \\ \dfrac{\partial \theta}{\partial t} + u_i\dfrac{\partial \theta}{\partial x_i} + w_i\dfrac{\partial S}{\partial x_i} = \kappa\Delta\theta, \end{cases} \tag{4.9}$$

$$\begin{cases} \dfrac{\mu}{k}w_i^m = -\dfrac{\partial \pi^m}{\partial x_i} + g_i\theta^m, \\ \dfrac{\partial w_i^m}{\partial x_i} = 0, \\ \dfrac{\partial \theta^m}{\partial t} + u_i^m\dfrac{\partial \theta^m}{\partial x_i} + w_i^m\dfrac{\partial S^m}{\partial x_i} = \kappa^m\Delta\theta^m, \end{cases} \tag{4.10}$$

in the spatial domains D_1 and D_2, respectively. The initial conditions are

$$\begin{aligned} &w_i(x,0) = 0, \qquad \theta(x,0) = 0, \quad x \in D_1, \\ &\theta^m(x,0) = 0, \quad x \in D_2, \end{aligned} \tag{4.11}$$

while the boundary conditions on $\Gamma_1 \cup \Gamma_2$ are

$$\begin{cases} w_i = 0, \quad \theta = 0, & \text{on} \quad \Gamma_1 \times (0, \mathcal{T}), \\ w_i^m n_i = 0, \quad \theta^m = 0, & \text{on} \quad \Gamma_2 \times (0, \mathcal{T}). \end{cases} \tag{4.12}$$

The conditions satisfied on the interface L become

$$\begin{cases} w_3 = w_3^m, \qquad \theta = \theta^m, \qquad \kappa\theta_{,3} = \kappa^m\theta^m_{,3}, \\ \pi^m = \pi - 2\mu w_{3,3}, \qquad w_{\beta,3} + w_{3,\beta} = \dfrac{\alpha_1}{\sqrt{k}}w_\beta + \dfrac{\sigma}{\sqrt{k}}v_\beta. \end{cases} \tag{4.13}$$

We now establish the following theorem which demonstrates continuous dependence of a solution on the interface coefficient α_1.

THEOREM 3. – *Suppose* $\partial T/\partial n \in L^1(\Gamma_1 \times (0,\mathcal{T}))$ *and* $\partial T^m/\partial n \in L^1(\Gamma_2 \times (0,\mathcal{T}))$. *Then there exist constants* $\gamma(< 2\mu/k)$, B, C *and* $\hat{A}$, *given by* (4.26) *and* (4.32), *such that*

$$\begin{aligned}(4.14)\qquad &\int_{D_1} w_i w_i\,dx + B\int_0^t\int_{D_1} w_i w_i\,dx\,d\eta + \gamma\int_0^t\int_{D_2} w_i^m w_i^m\,dx\,d\eta\\ &\leq \frac{Ce^{Bt}}{\alpha_1\alpha_2}\left(\int_{D_1} f_i f_i\,dx + \hat{A}tT_m^2\right)\sigma^2.\end{aligned}$$

Furthermore, there is a constant M, *depending on* t, *given specifically in* (4.33), *such that*

$$(4.15)\qquad \int_{D_1}\theta^2 dx + \int_{D_2}(\theta^m)^2 dx \leq \frac{M}{\alpha_1\alpha_2}\sigma^2.$$

Proof. – Let us introduce the notation of norm and inner product on the spaces $L^2(D_1)$ and $L^2(D_2)$, viz.

$$\|f\|_\alpha^2 = \int_{D_\alpha} f_i f_i\,dx\,,\qquad (f,g)_\alpha = \int_{D_\alpha} f_i g_i\,dx\,,\qquad \alpha = 1,2.$$

We then define the function $\Phi(t)$ by

$$(4.16)\qquad \Phi(t) = \|\mathbf{w}\|_1^2 + \gamma\int_0^t \|\mathbf{w}^m\|_2^2 d\eta.$$

By calculation we obtain

$$\begin{aligned}(4.17)\qquad \frac{d\Phi}{dt} &= 2(w_i, w_{i,t})_1 + \gamma\|\mathbf{w}^m\|_2^2\\ &= 2\Big(w_i, \big[-\pi_{,i} + \mu(w_{i,j}+w_{j,i})_{,j} + g_i\theta\big]\Big)_1 + \gamma\|\mathbf{w}^m\|_2^2,\end{aligned}$$

where we have used (4.9) and added a zero term. The first two terms on the right are integrated by parts to find

$$\begin{aligned}(4.18)\qquad & -2(w_i,\pi_{,i})_1 + 2\mu\Big(w_i, [w_{i,j}+w_{j,i}]_{,j}\Big)_1\\ &= -2\int_L n_i^{(1)} w_i\pi\,dA + 2\mu\int_L n_j^{(1)} w_i(w_{i,j}+w_{j,i})dA - 2\mu(w_{i,j}, w_{i,j}+w_{j,i})_1\\ &= 2\int_L n_3^{(2)} w_3(\pi - 2\mu w_{3,3})\,dA\\ &\quad - 2\mu\int_L n_3^{(2)} w_\beta(w_{\beta,3}+w_{3,\beta})dA - 2\mu(w_{i,j}, w_{i,j}+w_{j,i})_1\\ &= 2\int_L n_i^{(2)} w_i^m\pi^m\,dA - 2\mu\int_L n_3^{(2)} w_\beta\left(\frac{\alpha_1}{\sqrt{k}}w_\beta + \frac{\sigma}{\sqrt{k}}v_\beta\right)dA\\ &\quad - 2\mu(w_{i,j}, w_{i,j}+w_{j,i})_1\,,\end{aligned}$$

where the interface conditions (4.13) have also been employed. Next use the divergence theorem on the first term on the right in (4.18) and then use equations (4.10) to find:

$$(4.19)\quad -2(w_i,\pi_{,i})_1+2\mu\big(w_i,[w_{i,j}+w_{j,i}]_{,j}\big)_1=2\Big(w_i^m,g_i\theta^m-\frac{\mu}{k}\,w_i^m\Big)_2 - 2\mu\int_L n_3^{(2)}w_\beta\Big(\frac{\alpha_1}{\sqrt{k}}\,w_\beta+\frac{\sigma}{\sqrt{k}}\,v_\beta\Big)dA - 2\mu(w_{i,j},w_{i,j}+w_{j,i})_1\,.$$

Thus, employing (4.19) in (4.17) we may derive

$$\frac{d\Phi}{dt}=-\Big(2\frac{\mu}{k}-\gamma\Big)\|\mathbf{w}^m\|_2^2-2\mu\int_L\Big(\frac{\alpha_1}{\sqrt{k}}\,w_\beta w_\beta+\frac{\sigma}{\sqrt{k}}\,w_\beta v_\beta\Big)dA - 2\mu(w_{i,j},w_{i,j}+w_{j,i})_1+2g_i(\theta,w_i)_1+2g_i(\theta^m,w_i^m)_2.$$

We next use the arithmetic-geometric mean inequality on the interface term involving $w_\beta v_\beta$ and on the last two terms. Use is also made of the Poincaré-like inequality

$$(4.20)\qquad \lambda\|\mathbf{w}\|_1^2\le(w_{i,j},w_{i,j}+w_{j,i})_1$$

a proof of which is given after (4.34) for completeness. Thus for $\alpha,\zeta>0$ at our disposal we derive

$$\frac{d\Phi}{dt}\le-\Big(\frac{2\mu}{k}-\gamma-\zeta\Big)\|\mathbf{w}^m\|_2^2+\frac{\mu}{2\alpha_1\sqrt{k}}\,\sigma^2\int_L v_\beta v_\beta dA - (2\mu\lambda-\alpha)\|\mathbf{w}\|_1^2+\alpha^{-1}\|\theta\|_1^2+\zeta^{-1}\|\theta^m\|_2^2.$$

Select now $\alpha=2\mu\lambda$ and choose $\gamma<2\mu/k$. Then pick

$$\zeta=\frac{2\mu}{k}-\gamma>0.$$

We thus arrive at the differential inequality

$$(4.21)\qquad \frac{d\Phi}{dt}\le\frac{\mu}{2\alpha_1\sqrt{k}}\,\sigma^2\int_L v_\beta v_\beta dA+\frac{1}{2\mu\lambda}\|\theta\|_1^2+\zeta^{-1}\|\theta^m\|_2^2.$$

An estimate for $\|\theta\|_1^2$ *and* $\|\theta^m\|_2^2$. By differentiation and substitution from equations (4.9) and (4.10) we see that

$$(4.22)\qquad \frac{d}{dt}\big(\|\theta\|_1^2+\|\theta^m\|_2^2\big)=2\big(\theta,[-u_i\theta_{,i}-w_iS_{,i}+\kappa\Delta\theta]\big)_1 + 2\big(\theta^m,[-u_i^m\theta_{,i}^m-w_i^mS_{,i}^m+\kappa^m\Delta\theta^m]\big)_2.$$

Next, use integration and the boundary conditions (4.12), (4.13) to show that

$$(\theta, u_i\theta_{,i})_1 + (\theta^m, u_i^m\theta_{,i}^m)_2 = 0.$$

Furthermore, by integration by parts and further use of (4.12) and (4.13) we may then deduce from (4.22)

$$\frac{d}{dt}(\|\theta\|_1^2 + \|\theta^m\|_2^2) = 2\int_{D_1} w_i\theta_{,i}S\,dx + 2\int_{D_2} w_i^m\theta_{,i}^m S^m\,dx - 2\kappa\|\nabla\theta\|_1^2 - 2\kappa^m\|\nabla\theta^m\|_2^2. \tag{4.23}$$

To proceed from this equation we estimate S and S^m.

An estimate for T, T^m. Although we derive an estimate for T and T^m the proof clearly holds also for S and S^m.

For $p > 1$, $p \in \mathbf{N}$, p even, introduce the function

$$\Phi_p = \int_{D_1} T^p dx + \int_{D_2} (T^m)^p dx.$$

Then upon differentiation and use of equations (4.1) and (4.2),

$$\begin{aligned}
\frac{d\Phi_p}{dt} &= p\int_{D_1} T^{p-1}T_{,t}dx + p\int_{D_2} (T^m)^{p-1}T_{,t}^m dx\\
&= p\int_{D_1} T^{p-1}(-u_iT_{,i} + \kappa\Delta T)dx + p\int_{D_2} (T^m)^{p-1}(-u_i^mT_{,i}^m + \kappa^m\Delta T^m)dx\\
&= -p\oint_{\partial D_1} T^p u_i n_i^{(1)}dS + \kappa p\oint_{\partial D_1} T^{p-1}\frac{\partial T}{\partial n}\,dS - \kappa p(p-1)\int_{D_1} T^{p-2}T_{,i}T_{,i}dx\\
&\quad - p\oint_{\partial D_2} (T^m)^p u_i^m n_i^{(2)}dS + \kappa^m p\oint_{\partial D_2} (T^m)^{p-1}\frac{\partial T^m}{\partial n}\,dS\\
&\quad - \kappa^m p(p-1)\int_{D_2} (T^m)^{p-2}T_{,i}^mT_{,i}^m dx,
\end{aligned}$$

where ∂D_1 and ∂D_2 denote the boundaries $\Gamma_1 \cup L$ and $\Gamma_2 \cup L$, respectively. Due to the continuity conditions (4.5) the integrals over L vanish and thus we may derive the inequality:

$$\begin{aligned}
\frac{d\Phi_p}{dt} &\leq \kappa p\int_{\Gamma_1} T^{p-1}\frac{\partial T}{\partial n}\,dS + \kappa^m p\int_{\Gamma_2} (T^m)^{p-1}\frac{\partial T^m}{\partial n}\,dS\\
&\leq \kappa p|T^{p-1}|_m\int_{\Gamma_1}\left|\frac{\partial T}{\partial n}\right| dS + \kappa^m p|T^m|_m^{p-1}\int_{\Gamma_2}\left|\frac{\partial T^m}{\partial n}\right| dS,
\end{aligned}$$

where $|\cdot|_m$ denotes the maximum value on $\Gamma_\alpha \times (0, \mathcal{T})$. After integration, we obtain:

$$\begin{aligned}
\Phi_p^{1/p}(t) \leq \Bigg[\Phi_p(0) &+ \kappa p|T^{p-1}|_m\int_0^t\int_{\Gamma_1}\left|\frac{\partial T}{\partial n}\right| dS\,d\eta\\
&+ \kappa^m p|T^m|_m^{p-1}\int_0^t\int_{\Gamma_2}\left|\frac{\partial T^m}{\partial n}\right| dS\,d\eta\Bigg]^{1/p}.
\end{aligned}$$

By taking the limit $p \to \infty$ we thus derive

$$|T| \le T_m = \max\{|T_0|_m, |T_0^m|_m, |T_U|_m, |T_L|_m\}, \tag{4.24}$$

where on the right $|\cdot|_m$ denotes the maximum over the respective domain and T_m is a bound for the maximum value of $|T|$ on $D_1 \cup D_2$.

Thus, we may return to (4.23) and bound S and S^m by T_m. We do this and use the arithmetic-geometric mean inequality on the $(w_i, \theta_{,i})_1$ and $(w_i^m, \theta_{,i}^m)_2$ terms to derive the inequality

$$\frac{d}{dt}\left(\|\theta\|_1^2 + \|\theta^m\|_2^2\right) \le \frac{T_m^2}{2\kappa}\|\mathbf{w}\|_1^2 + \frac{T_m^2}{2\kappa^m}\|\mathbf{w}^m\|_2^2.$$

Upon integration we thus deduce

$$\|\theta(t)\|_1^2 + \|\theta^m(t)\|_2^2 \le \frac{T_m^2}{2\kappa}\int_0^t \|\mathbf{w}\|_1^2 d\eta + \frac{T_m^2}{2\kappa^m}\int_0^t \|\mathbf{w}^m\|_2^2 d\eta. \tag{4.25}$$

This is the required bound for $\|\theta\|_1$ and $\|\theta^m\|_2$.

Define now the constant A by

$$A = \frac{1}{2}T_m^2 \max\left\{\frac{1}{2\mu\lambda}, \zeta^{-1}\right\}.$$

Further, define Φ_1 and Φ_2 by

$$\Phi_1(t) = \|\mathbf{w}\|_1^2, \qquad \Phi_2(t) = \int_0^t \|\mathbf{w}^m\|_2^2 d\eta.$$

From inequality (4.21) we now observe that

$$\frac{d}{dt}(\Phi_1 + \gamma\Phi_2) \le \left(\frac{1}{\kappa}\int_0^t \Phi_1 d\eta + \frac{1}{\kappa^m}\Phi_2\right) + \frac{\mu}{2\alpha_1\sqrt{k}}\sigma^2 \int_L v_\beta v_\beta \, dA. \tag{4.26}$$

Define now B to be the constant

$$B = \max\left\{\sqrt{\frac{A}{\kappa}}, \frac{A}{\kappa^m \gamma}\right\}. \tag{4.27}$$

Further, define $\Psi_1(t)$ to be the function

$$\Psi_1(t) = \int_0^t \Phi_1(\eta) d\eta.$$

From (4.26) one may then deduce

$$\frac{d^2\Psi_1}{dt^2} + \gamma\frac{d\Phi_2}{dt} \le B^2\Psi_1 + \gamma B\Phi_2 + \frac{\mu}{2\alpha_1\sqrt{k}}\sigma^2 \int_L v_\beta v_\beta dA. \tag{4.28}$$

This inequality may be rearranged to yield

$$\left(\frac{d}{dt}-B\right)\left\{\frac{d\Psi_1}{dt}+B\Psi_1+\gamma\Phi_2\right\}\le\frac{\mu}{2\alpha_1\sqrt{k}}\sigma^2\int_L v_\beta v_\beta dA. \tag{4.29}$$

Observe that $d\Psi_1/dt=\Phi_1\ge 0$ so the function on the left of (4.29) operated on by $(d/dt-B)$ is non-negative. Thus after integration we may derive from (4.29) the estimate:
(4.30)

$$\|\mathbf{w}\|_1^2+B\int_0^t\|\mathbf{w}\|_1^2d\eta+\gamma\int_0^t\|\mathbf{w}^m\|_2^2\,d\eta\le\left(\frac{\mu}{2\alpha_1\sqrt{k}}\int_0^t\int_L e^{B(t-\eta)}v_\beta v_\beta dA\,d\eta\right)\sigma^2.$$

A bound for the quantity $\int_L v_\beta v_\beta dA$. To complete the continuous dependence analysis it remains to derive a bound for the right hand side of (4.30). To this end we return to equations (4.1), (4.2) and derive

$$\begin{aligned}\frac{d}{dt}\|\mathbf{v}\|_1^2=&-2(v_i,q_{,i})_1+2\mu\big(v_i,(v_{i,j}+v_{j,i})_{,j}\big)_1+2(v_i,g_iS)_1\\ =&-2\int_L n_3^{(1)}v_3(q-2\mu v_{3,3})dA+2\mu\int_L n_3^{(1)}v_\beta(v_{\beta,3}+v_{3,\beta})dA\\ &-2\mu(v_{i,j},v_{i,j}+v_{j,i})_1+2(v_i,g_iS)_1\\ =&\,2\int_L n_3^{(2)}v_3^m q^m dA-2\mu\int_L v_\beta(v_{\beta,3}+v_{3,\beta})dA\\ &-2\mu(v_{i,j},v_{i,j}+v_{j,i})_1+2(v_i,g_iS)_1\\ =&-\frac{2\mu\alpha_2}{\sqrt{k}}\int_L v_\beta v_\beta dA-\frac{2\mu}{k}\|\mathbf{v}^m\|_2^2-2\mu(v_{i,j},v_{i,j}+v_{j,i})_1\\ &+2(v_i,g_iS)_1+2(v_i^m,g_iS^m)_2\,,\end{aligned} \tag{4.31}$$

where we have followed the procedure after (4.17). Now, noting S and S^m are bounded by T_m we may use the arithmetic-geometric mean inequality on the last two terms in (4.31) together with the Poincaré-like inequality for $\mathbf{v}$,

$$\lambda\|\mathbf{v}\|_1^2\le(v_{i,j},v_{i,j}+v_{j,i})_1.$$

From (4.31) we may thus deduce

$$\frac{d}{dt}\|\mathbf{v}\|_1^2\le-\frac{2\mu\alpha_2}{\sqrt{k}}\int_L v_\beta v_\beta dA+\frac{T_m^2}{2\mu}\left(\frac{|D_1|}{\lambda}+|D_2|k\right),$$

where $|D_\alpha|$ is the measure of the domain D_α. Upon integration we now find

$$\|\mathbf{v}(t)\|_1^2+\frac{2\mu\alpha_2}{\sqrt{k}}\int_0^t\int_L v_\beta v_\beta dA\le\|\mathbf{v}(0)\|_1^2+\frac{T_m^2}{2\mu}\left(\frac{|D_1|}{\lambda}+|D_2|k\right)t. \tag{4.32}$$

Upon employing the bound (4.32) in (4.30) we, therefore, derive the continuous dependence inequality

$$\begin{aligned}(4.33)\qquad &\|\mathbf{w}\|_1^2 + B\int_0^t \|\mathbf{w}\|_1^2 d\eta + \gamma\int_0^t \|\mathbf{w}^m\|_2^2\, d\eta \\ &\le \frac{e^{Bt}}{4\alpha_1\alpha_2}\left[\|\mathbf{f}\|_1^2 + \frac{T_m^2}{2\mu}\left(\frac{|D_1|}{\lambda} + |D_2|k\right)t\right]\sigma^2.\end{aligned}$$

To produce a continuous dependence estimate for θ amd θ^m we may utilize (4.33) and employ (4.25) to see that

$$(4.34)\qquad \|\theta(t)\|_1^2 + \|\theta^m(t)\|_2^2 \le \frac{\hat{\kappa}T_m^2 e^{Bt}}{8\alpha_1\alpha_2}\left[\|\mathbf{f}\|_1^2 + \frac{T_m^2}{2\mu}\left(\frac{|D_1|}{\lambda} + |D_2|k\right)t\right]\sigma^2.$$

Inequalities (4.33) and (4.34) establish Theorem 3. However, it remains to prove the Poincaré-like inequality (4.20). We could establish the existence of a maximising solution to the variational problem associated with (4.20). However, we actually need an estimate for the constant λ in (4.20) and hence we derive such an estimate directly.

A proof of the Poincaré-like inequality (4.20). Let D_1 be the domain as in (4.1) and let w_i be an arbitrary solenoidal C^1 function satisfying the boundary conditions $w_i = 0$ on Γ_1. Suppose the coordinate system is defined such that L lies in the plane $z = x_3 = 0$. Suppose further d_1 is the maximum distance from the boundary of Γ_1 to $(0,0,z)$ for any $z \in D_1$, and let d_2 denote the maximum distance from $(0,0,0)$ to Γ_1 (to the outer boundary of Γ_1 if D_1 is star-shaped with respect to the origin).

For a C^1 function f_i to be chosen observe that

$$\oint_{\partial D_1} f_i n_i w_j w_j\, dS = \int_{D_1} f_{i,i} w_j w_j\, dx + 2\int_{D_1} f_i w_j w_{j,i}\, dx$$

and

$$\oint_{\partial D_1} f_i n_j w_i w_j\, dS = \int_{D_1} f_{i,j} w_i w_j\, dx + \int_{D_1} f_i w_j w_{i,j}\, dx,$$

where $\partial D_1 = \Gamma_1 \cup L$ and $n_i = n_i^{(1)}$. Then, by suitably adding these equations we may deduce

$$\begin{aligned}(4.35)\qquad &\oint_{\partial D_1} f_i n_i w_j w_j\, dS + 2\oint_{\partial D_1} f_i n_j w_i w_j\, dS - \int_{D_1} f_{i,i} w_j w_j\, dx \\ &- 2\int_{D_1} f_{i,j} w_i w_j\, dx = 2\int_{D_1} f_i w_j (w_{i,j} + w_{j,i})\, dx.\end{aligned}$$

Select now f_i such that

$$f_\alpha = -2\nu x_\alpha, \quad \alpha = 1,2; \qquad f_3 = -\xi x_3 - A,$$

for positive constants ν, ξ, A to be chosen. Then, recalling $w_i = 0$ on Γ_1 we see from (4.35) that

$$(4.36)\qquad \begin{aligned} &3A\int_L w_3^2\,dA + A\int_L w_\beta w_\beta\,dA + 4\nu\int_L x_\alpha w_\alpha w_3\,dA \\ &\quad + (4\nu+3\xi)\int_{D_1} w_3^2\,dx + (8\nu+\xi)\int_{D_1} w_\alpha w_\alpha\,dx \\ &\quad = 2\int_{D_1} f_i w_j(w_{i,j}+w_{j,i})dx. \end{aligned}$$

Now choose $\xi = 2\nu$ in (4.36) and observe from the arithmetic-geometric mean inequality

$$4\nu\left|\int_L x_\alpha w_\alpha w_3\,dA\right| \le 2\nu d_1\left(\sqrt{3}\int_L w_3^2\,dA + \frac{1}{\sqrt{3}}\int_L w_\alpha w_\alpha\,dA\right).$$

Hence, upon employing this inequality in (4.36) and using the Cauchy-Schwarz inequality on the right hand side we find that:

$$(4.37)\qquad \begin{aligned} &\left(A - \frac{2\nu d_1}{\sqrt{3}}\right)\left(3\int_L w_3^2\,dA + \int_L w_\alpha w_\alpha\,dA\right) + 10\nu\int_{D_1} w_i w_i\,dx \\ &\quad \le 2^{3/2}\sqrt{\int_{D_1} f_i f_i w_j w_j\,dx}\sqrt{\int_{D_1} w_{i,j}(w_{i,j}+w_{j,i})\,dx}. \end{aligned}$$

Now, we have:

$$(4.38)\qquad \begin{aligned} f_i f_i &= 4\nu^2 x_\alpha x_\alpha + (A+\xi x_3)^2 \\ &= 4\nu^2 x_\alpha x_\alpha + A^2 + 4\nu A x_3 + 4\nu^2 x_3^2 \\ &\le 4\nu^2 x_i x_i + A^2 + 4\nu A|x_i| \\ &\le (A+2\nu d_2)^2, \end{aligned}$$

since $d_1 \le d_2$. Choose now $A = 2\nu d_2/\sqrt{3}$ and then employing (4.38) in (4.37) we derive the inequality

$$(4.39)\qquad \frac{75}{8d_2^2(1+\sqrt{3})^2}\,\|\mathbf{w}\|_1^2 \le \int_{D_1} w_{i,j}(w_{i,j}+w_{j,i})\,dx.$$

Thus $\hat{\lambda} = 75/8d_2^2(1+\sqrt{3})^2$ represents a lower bound for the value of $\lambda(D_1)$ in (4.20).

Remark. – Suppose one wishes to extend the analysis of this section to the case where the fluid in the domain satisfies the Navier-Stokes equations, *i.e.* $(4.1)_1$ includes also the term $u_j u_{i,j}$ on the left hand side. Then in trying to derive an expression like (4.18) one encounters the term

$$-2(w_i, \pi_{,i})_1 + 2\mu\big(w_i, [w_{i,j}+w_{j,i}]_{,j}\big)_1 - 2\big(w_i, (u_j u_i)_{,j} - (v_j v_i)_{,j}\big)_1.$$

This would give rise to terms in the integral over L of form

$$-2\int_L n_i^{(1)} w_i \pi \, dA + 2\mu \int_L n_j^{(1)} w_i [w_{i,j} + w_{j,i}] \, dA$$
$$-2\int_L n_j^{(1)} w_i (u_j u_i - v_j v_i) \, dA.$$

To handle the last term one may wish to modify the continuity of normal stress condition $(4.5)_4$, or perhaps a suitable modification of the Beavers-Joseph-Jones condition $(4.5)_5$ would be necessary. In any case, this analysis suggests that for flow of a Navier-Stokes fluid past a porous medium the conditions (4.5) may not be adequate.

Appendix

In remark 1 in section 3 we indicated the restriction $\partial T/\partial n \in L^1(\partial\Omega \times (0, \mathcal{T}))$ is only necessary to produce inequality (3.29). We now indicate a way to circumvent this restriction and produce a bound like (2.18).

Let T and v_i be as in section 3. Then define $\mathcal{H}$ to be the solution to the boundary value problem

$$\text{(A1)} \qquad \begin{cases} \Delta \mathcal{H} = 0, & \text{in } \Omega, \\ \mathcal{H} = h^{2p-1}, & \text{on } \partial\Omega, \end{cases}$$

for $t \in [0, \mathcal{T}]$ and for $p > 0$ fixed; it is sufficient that $p \in \mathbf{N}$. Consider now the identity

$$\text{(A2)} \qquad \int_0^t \int_\Omega (T^{2p-1} - \mathcal{H})(T_{,t} + v_i T_{,i} - \Delta T) dx\, d\eta = 0.$$

(The temperature field may be chosen positive, this amounts to selecting an appropriate temperature scale.)

After integration by parts in (A2) and some rearrangement we may produce

$$\text{(A3)} \qquad \int_\Omega T^{2p} dx + \frac{2(2p-1)}{p} \int_0^t \int_\Omega T^p_{,i} T^p_{,i} dx\, d\eta = \int_\Omega T_0^{2p} dx + 2p(\mathcal{H}, T)$$
$$- 2p(\mathcal{H}_0, T_0) - 2p \int_0^t (\mathcal{H}_{,\eta}, T) d\eta$$
$$+ 2p \int_0^t \int_\Omega \mathcal{H} v_i T_{,i} \, dx\, d\eta + 2p \int_0^t \oint_{\partial\Omega} \frac{\partial \mathcal{H}}{\partial n} h \, dA\, d\eta.$$

To handle the cubic term we first note that from (3.1), (3.3) one easily derives inequality (2.35), namely

$$\text{(A4)} \qquad \|\mathbf{v}\| \le \|T\|.$$

Moreover, using the maximum principle

$$2p\int_0^t\int_\Omega \mathcal{H}v_iT_{,i}dx\,d\eta \le 2p\mathcal{H}_m\sqrt{\int_0^t \|\mathbf{v}\|^2d\eta\int_0^t \|\nabla T\|^2d\eta}\,, \tag{A5}$$

where $\mathcal{H}_m$ is the maximum value of $\mathcal{H}$ on $\partial\Omega\times[0,\mathcal{T}]$. Thus, combining (A4) and (A5),

$$2p\int_0^t\int_\Omega \mathcal{H}v_iT_{,i}dx\,d\eta \le 2ph_m^{2p-1}\sqrt{\int_0^t \|T\|^2d\eta\int_0^t \|\nabla T\|^2d\eta}. \tag{A6}$$

Hence, from (A3) we deduce

$$\begin{aligned}\int_\Omega T^{2p}dx \le{} & \int_\Omega T_0^{2p}dx + 2p\big(\|\mathcal{H}\|\|T\| + \|\mathcal{H}_0\|\|T_0\|\big)\\ & + 2p\sqrt{\int_0^t \|\mathcal{H}_{,\eta}\|^2d\eta\int_0^t \|T\|^2d\eta}\\ & + 2ph_m^{2p-1}\sqrt{\int_0^t \|T\|^2d\eta\int_0^t \|\nabla T\|^2d\eta}\\ & + 2p\sqrt{\int_0^t\oint_{\partial\Omega} h^2dA\,d\eta\int_0^t\oint_{\partial\Omega}\left(\frac{\partial\mathcal{H}}{\partial n}\right)^2 dA\,d\eta}\,.\end{aligned} \tag{A7}$$

Estimates for the terms on the right of (A7) are now derived.

Firstly, with G defined as in (2.29) one may integrate the identity analogous to (2.51) to find that, with some manipulation,

$$\begin{aligned}\|T\|^2 + 2\int_0^t \|\nabla T\|^2d\eta \le{} & 2\|T_0\|^2 + \frac{1}{2}\|T\|^2 + 2\|G\|^2\\ & + \|G_0\|^2 + 2\int_0^t \|G_{,\eta}\|\|T\|d\eta\\ & + 2h_m\int_0^t \|\mathbf{v}\|\|\nabla T\|d\eta\\ & + 2\int_0^t\sqrt{\oint_{\partial\Omega} h^2dA\oint_{\partial\Omega}\left(\frac{\partial G}{\partial n}\right)^2 dA}\,d\eta\,.\end{aligned} \tag{A8}$$

(Equivalently one may follow the development of (A2) with $p=1$.)

For a harmonic function ϕ with boundary values Q, *i.e.*

$$\begin{cases}\Delta\phi = 0, & \text{in } \Omega,\\ \phi = Q, & \text{on } \partial\Omega,\end{cases} \tag{A9}$$

we may use a Rellich identity to derive explicit positive constants c_1, c_2 such that

$$\text{(A10)} \qquad \|\nabla\phi\|^2 + c_1 \oint_{\partial\Omega} \left(\frac{\partial\phi}{\partial n}\right)^2 dA \le c_2 \oint_{\partial\Omega} |\nabla_s Q|^2 dA,$$

where ∇_s denotes the tangential derivative, *cf.* Payne & Weinberger (1958), Franchi & Straughan (1994b).

Next introduce the torsion-like function ψ which satisfies the boundary value problem

$$\text{(A11)} \qquad \begin{cases} \Delta\psi = -1, & \text{in } \Omega, \\ \psi = 0, & \text{on } \partial\Omega, \end{cases}$$

and observe $\psi > 0$ in Ω. Then for a function ϕ satisfying (A9), we have:

$$\text{(A12)} \qquad \begin{aligned} 2(\psi\nabla\phi, \nabla\phi) + \|\phi\|^2 &= -\oint_{\partial\Omega} \phi^2 \frac{\partial\psi}{\partial n} dA, \\ &\le \psi_1 \oint_{\partial\Omega} Q^2 dA, \end{aligned}$$

where

$$\text{(A13)} \qquad \psi_1 = \max_{\partial\Omega} \left|\frac{\partial\psi}{\partial n}\right|.$$

Application of (A12) yields bounds for $\|G\|^2$ and $\|G_{,t}\|^2$ in terms of $L^2(\partial\Omega)$ integrals of their boundary data.

The inequality (A4) is next used on the sixth term on the right of (A8) and then with the aid of the arithmetic-geometric mean inequality one sees that

$$\text{(A14)} \qquad \begin{aligned} &2\int_0^t \|G_{,\eta}\|\|T\| d\eta + 2h_m \int_0^t \|T\|\|\nabla T\| d\eta \\ &\le \int_0^t \|G_{,\eta}\|^2 d\eta + \int_0^t \|\nabla T\|^2 d\eta + (1 + h_m^2) \int_0^t \|T\|^2 d\eta. \end{aligned}$$

By employing (A14) in (A8) together with inequalities (A10) and (A12), we thus derive

$$\text{(A15)} \qquad \begin{aligned} \frac{1}{2}\|T\|^2 + \int_0^t \|\nabla T\|^2 d\eta &\le 2\|T_0\|^2 + 2\|G\|^2 + \|G_0\|^2 \\ &\quad + 2\sqrt{\frac{c_2}{c_1}} \int_0^t \left(\oint_{\partial\Omega} h^2 dA \oint_{\partial\Omega} |\nabla_s h|^2 dA\right)^{1/2} d\eta \\ &\quad + \int_0^t \|G_{,\eta}\|^2 d\eta + (1 + h_m^2) \int_0^t \|T\|^2 d\eta, \\ &\le 2\|T_0\|^2 + 3\psi_1 \oint_{\partial\Omega} h^2 dA \\ &\quad + 2\sqrt{\frac{c_2}{c_1}} \int_0^t \left(\oint_{\partial\Omega} h^2 dA \oint_{\partial\Omega} |\nabla_s h|^2 dA\right)^{1/2} d\eta \\ &\quad + \psi_1 \int_0^t \oint_{\partial\Omega} h^2 dA\, d\eta + (1 + h_m^2) \int_0^t \|T\|^2 d\eta. \end{aligned}$$

Next, set

$$
\text{(A16)} \quad d_1 = \max_{[0,T]} \left\{ 2\|T_0\|^2 + 3\psi_1 \oint_{\partial\Omega} h^2 dA + 2\sqrt{\frac{c_2}{c_1}} \int_0^t \sqrt{\oint_{\partial\Omega} h^2 dA \oint_{\partial\Omega} |\nabla_s h|^2 dA}\, d\eta \right.
$$
$$
\left. + \psi_1 \int_0^t \oint_{\partial\Omega} h^2 dA\, d\eta \right\}.
$$

Then use of (A16) in (A15) leads to

$$
\text{(A17)} \qquad \|T\|^2 + 2\int_0^t \|\nabla T\|^2 d\eta \le 2d_1 + \zeta \int_0^t \|T\|^2 d\eta,
$$

where we have set $\zeta = 2(1 + h_m^2)$. From inequality (A17) we easily deduce

$$
\frac{d}{dt}\left(e^{-\zeta t} \int_0^t \|T\|^2 d\eta \right) \le 2d_1 e^{-\zeta t},
$$

and then we may show

$$
\text{(A18)} \qquad \int_0^t \|T\|^2 d\eta \le \frac{2d_1}{\zeta}\,(e^{\zeta t} - 1),
$$

$$
\text{(A19)} \qquad \int_0^t \|\nabla T\|^2 d\eta \le d_1\, e^{\zeta t}
$$

and

$$
\text{(A20)} \qquad \|T\|^2 \le 2d_1\, e^{\zeta t}.
$$

We are now in a position to return to inequality (A7). We utilize (A18)-(A20) together with the following estimates which arise from (A12),

$$
\text{(A21)} \qquad \|\mathcal{H}\|^2 \le \psi_1 \oint_{\partial\Omega} h^{4p-2} dA,
$$

$$
\text{(A22)} \qquad \|\mathcal{H}_{,t}\|^2 \le \psi_1 \oint_{\partial\Omega} \left[(h^{2p-1})_{,t}\right]^2 dA,
$$

$$
\text{(A23)} \qquad \oint_{\partial\Omega} \left(\frac{\partial\mathcal{H}}{\partial n}\right)^2 dA \le \frac{c_2}{c_1} \oint_{\partial\Omega} \left|\nabla_s h^{2p-1}\right|^2 dA.
$$

In this manner we derive the estimate

$$\text{(A24)}\qquad \int_\Omega T^{2p}dx \le \int_\Omega T_0^{2p}dx + 2p\big(\sqrt{2d_1e^{\zeta\mathcal{T}}} + \|T_0\|\big)\psi_1^{1/2}\Big(\oint_{\partial\Omega} h^{4p-2}dA\Big)^{1/2}$$
$$+ 2p\sqrt{\frac{2d_1}{\zeta}\,(e^{\zeta\mathcal{T}}-1)}\Big(\int_0^t \psi_1 \oint_{\partial\Omega} h^{4p-4}h_{,t}^2\,dA\,d\eta\Big)^{1/2}$$
$$+ 2ph_m^{2p-1}\sqrt{2d_1^2e^{\zeta\mathcal{T}}\int_0^t e^{\zeta\mathcal{T}}\,d\eta}$$
$$+ 2p\sqrt{\frac{c_2}{c_1}}\Big(\int_0^t\oint_{\partial\Omega} h^2dA\,d\eta\Big)^{1/2}\Big(\int_0^t\oint_{\partial\Omega} h^{4p-4}|\nabla_s h|^2\,dA\,d\eta\Big)^{1/2}.$$

Then from (A24) we find:

$$\text{(A25)}\qquad \Big[\int_\Omega T^{2p}dx\Big]^{1/2p} \le \Big[\int_\Omega T_0^{2p}dx + \frac{2p\psi_1^{1/2}\big[m(\partial\Omega)\big]^{1/2}}{h_m}\big(\sqrt{2d_1e^{\zeta\mathcal{T}}} + \|T_0\|\big)h_m^{2p}$$
$$+ \frac{2p\psi_1^{1/2}k_1}{h_m^2}\sqrt{\frac{2d_1}{\zeta}\,(e^{\zeta\mathcal{T}}-1)}\Big(\int_0^t\oint_{\partial\Omega} h_{,\eta}^2\,dA\,d\eta\Big)^{1/2} h_m^{2p}$$
$$+ \frac{2^{3/2}pd_1e^{\zeta\mathcal{T}}\sqrt{t}}{h_m}\,h_m^{2p}$$
$$+ 2p\sqrt{\frac{c_2}{c_1}}\frac{k_1}{h_m^2}\Big(\int_0^t\oint_{\partial\Omega} h^2dA\,d\eta\Big)^{1/2}\Big(\int_0^t\oint_{\partial\Omega} |\nabla_s h|^2\,dA\,d\eta\Big)^{1/2} h_m^{2p}\Big]^{1/2p}.$$

Denote the coefficients of the second to fifth terms in parentheses, on the right of (A25), by $r_2(p)$ to $r_5(p)$; then

$$\text{(A26)}\qquad \|T\|_{2p} \le \Big[\|T_0\|_{2p}^{2p} + \Big\{r_2 + r_3 + r_4 + r_5\Big\}h_m^{2p}\Big]^{1/2p}.$$

Now let $2p \to \infty$ to obtain

$$\text{(A27)}\qquad \sup_{\Omega\times[0,\mathcal{T}]} |T| \le \max\Big\{|T_0|_m, \sup_{[0,\mathcal{T}]} h_m\Big\}.$$

This is the same estimate as (2.18) but obviates the need to restrict $\partial T/\partial n$ to the class $L^1(\partial\Omega \times (0,\mathcal{T}))$.

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